Mathematical Methods Units 3&4
ClassPad activities

Using technology to support mathematics learning

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Mathematical Methods Units 3&4 : ClassPad activities
Using technology to support mathematics learning

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Introduction

To the student

This book comprises a series of activities which are designed to facilitate learning the mathematics of the Mathematical Methods Units 3 & 4 course through the use of CAS technology as implemented on the ClassPad. The activities cover neither the whole course, nor are they restricted to purely course material.

Activities beyond the course content can assist you to solve problems within the course while also increasing your ability to explore broader mathematical questions. The Problems chapter in particular has activities to help explore the course content in more challenging situations. This book is about mathematics with detailed instructions on how the technology can be used.

The activities vary in the time needed to complete them. Some are primarily concerned with how to perform a particular technique within a ClassPad application, and some use the ClassPad output as the starting point. In others, the ClassPad is only a small part of the activity.

The activities are arranged into chapters matching the topics outlined in the Australian Curriculum. Within each topic the activities reflect a possible sequence of learning related to that topic. Many activities can be used as a precursor to formal teaching of the concept thereby encouraging a sense-making approach.

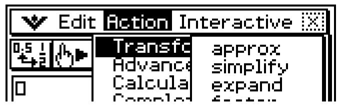
Each activity has an aim, linked to curriculum documents, the activity itself and usually a section of *Learning notes*. Fully worked solutions are provided at the end of the text. The Learning notes are intended to help with the understanding of concepts, provide more detail or help with instructions for ClassPad use, provide additional explanations or point to interesting further explorations. As the course progresses more assumptions are made about the skills you have developed and so the instructions become briefer. Where more detailed instructions are required on ClassPad use, it will often be in the *Learning notes* rather than in the text of the investigation.

Knowing when ClassPad use is quicker or more efficient becomes easier the more experience you have. Working through the activities will help you appreciate when use of ClassPad is more efficient.

What CAS enables us to do is to focus more on what to do rather than how to do it. For example, in a modelling situation we may come across awkward functions that we may not have the tools to deal with by traditional methods. Often, however, CAS enables you to get an answer so you then evaluate whether the result makes sense in the problem and thus demonstrate your understanding of the mathematics.

A lot of detail has been provided in the ClassPad instructions. However, it is impractical to cover all possible arrangements and settings. These activities were written for the ClassPad 400 series.

In the instructions:



- *Press* refers to a key on the ClassPad;
- *Select* and/or *[ ]* refers to a menu option, e.g. *[Action | Transformation | expand]*: The Action menu is at the top of the screen. Transformation is one of the options with expand an option in the submenu.

It is advisable to:

- check the settings such as Standard or Decimal, angles are in degrees, … Being familiar with options can save time;
- become familiar with the soft keyboard and where to find commands;
- clear previous working, [File |New] and [Edit |Clear All] may be helpful; and
- know how to clear variables and functions from Memory manager. If variables are stored from previous work that may lead to unexpected results. In particular if a variable has been used to define a function it is not cleared when clearing all variables.

The authors have mainly used the activities in class as an introduction to a topic or concept. During these periods we encourage students to talk and help each other. When students ask for help we can often best support them by asking questions like “Have you checked the Learning notes?” and “What do you think this means?”. Students are then able to work things out for themselves and more able to transfer the skills and concepts to new situations, a wonderful attribute when confronted with something novel in an assessment.

We hope you find these materials helpful, rewarding and of interest.

Ian Sheppard and Andrew Pateman

Chapter 1 Differential calculus and applications

Actvity	ClassPad applications	Key concepts
A function equal to its gradient	Graph&Table Main Spreadsheet Sequence	Explore and define a function that has itself as its derivative
Differentiating exponential functions	Spreadsheet Main	Differentiate a^x and establish the limit $\lim\limits_{h\to 0}\frac{a^h-1}{h}$
Route 2.7…*e*	Main Financial	Investigate continuously compounding interest, define *e* as a limit and connect to continuous growth
Growth and decay	Statistics Main	Model growth and decay situations using exponential functions
Differentiating trigonometric functions	Graph&Table Main	Investigate the derivative of the sine and cosine functions
The second derivative	Main	Use CAS and the second derivative test to determine nature of stationary points
Graphing functions	Graph&Table Main	Use calculus to determine key features of graphs
Composite functions	Graph&Table Main	Explore the composition of functions and associated domains and ranges
Gradient of composite functions	Main	Verify the chain rule
Pendulum motion	Statistics Main	Model periodic motion with trigonometric functions
Comfy chairs	Main	Optimisation with calculus
Silos'r'us	Main Spreadsheet	Use calculus to optimise dimensions of a silo for minimum cost

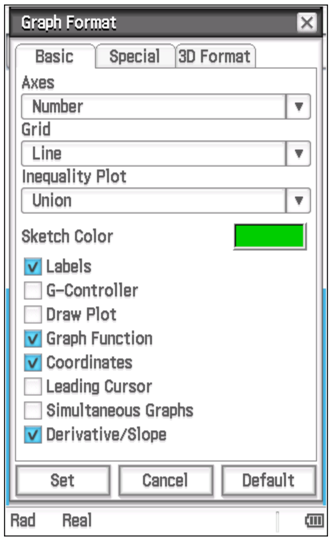
Activity 1 A function equal to its gradient

Aim: Explore and define a function that has itself as its derivative.

Is there a function where the value of the gradient is the same as the y-value throughout the domain? This will be explored graphically, numerically and algebraically in this activity.

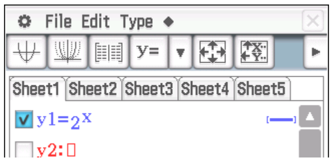
Setup

- Open Graph&Table from the Menu
- Tap
- Select Graph Format
- Tick the Derivative/Slope check box
- Tap Set



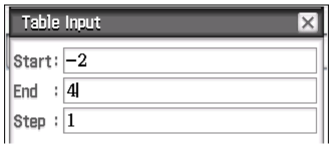
Enter the function $y = 2^x$

- Enter the function as shown (Use the keyboard as required)



Adjust the Table Input values

- Tap
- Set values as shown and tap OK.
- Tap to display the table.



1. Numerically

   a) Complete the table of values, rounded to 4 significant figures.

x	–2	–1	0	1	2	3	4
$y = 2^x$							
$\frac{dy}{dx}$							

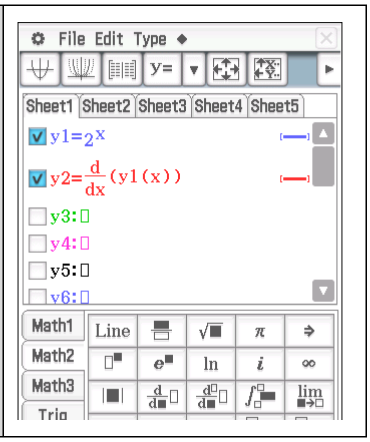
   b) What do you notice about the values of the derivative compared to the y-values?

2. Graphically

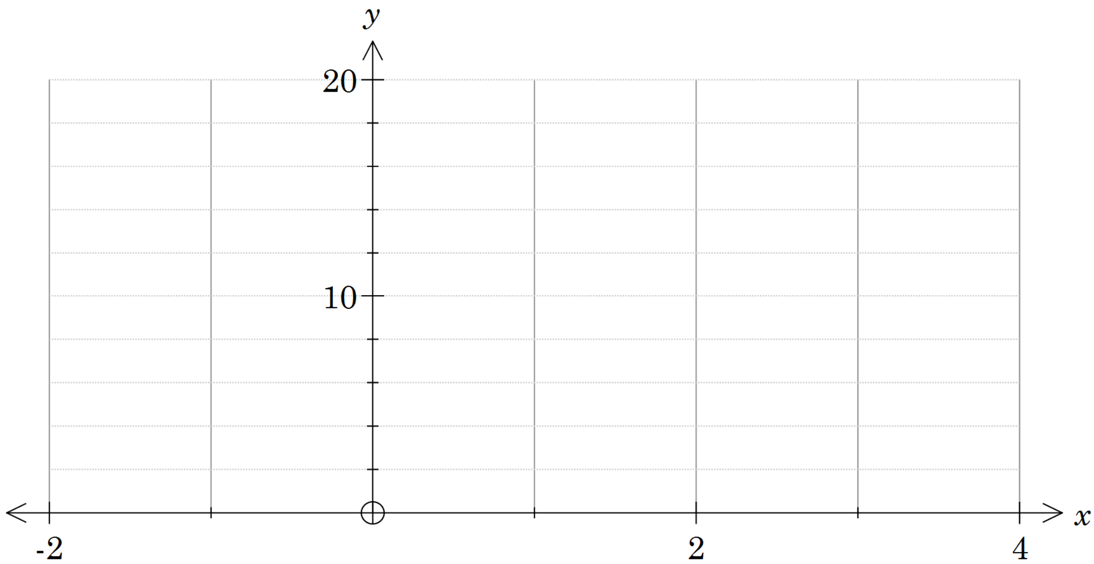
a) Graph $y = 2^x$ and its derivative.

Draw graph and derivative

- In Graph and Table add the derivative function as shown. (You can use the soft keyboard Math2 tab to access the derivative template $\frac{d}{d\blacksquare}\square$ and abc tab for y)
- Tap ⊞



Sketch both graphs on this grid.



b) What do you notice about the shape and vertical location of the derivative function compared to $y = 2^x$?

c) Change $y = 2^x$ to $y = 3^x$.

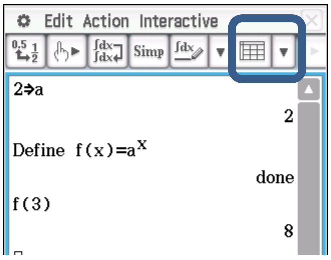
What do you notice about the shape and vertical location of the derivative function compared to $y = 3^x$?

d) Suggest a value for a in $y = a^x$ where the function and its derivative are closer together.

3. Build the spreadsheet.
The spreadsheet will be used to help look for a function with itself as derivative.

Define function in Main

- Store 2 as a
- Select [Interactive | Define]
  Set f(x)=a^x
- Check that your function is working, e.g. check that f(3)=8
- Open the Spreadsheet by selecting ▦ from the pull down menu
  Main and Spreadsheet will both be displayed



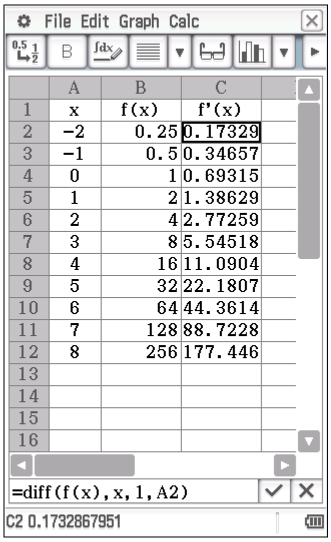
Build spreadsheet

Duplicate the spreadsheet as shown.
(See Learning notes for detailed instructions)
Formulae:

- A3: =A2+1
- B2: =f(A2)
- C2: =diff(f(x),x,1,A2)
- Fill down columns A, B and C

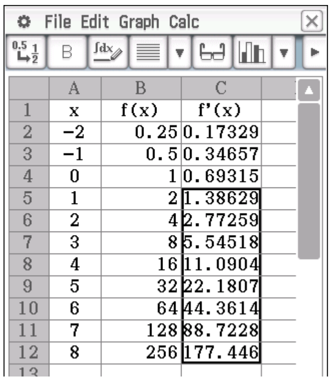
Save spreadsheet. (to use in a later activity)

- Select [File | Save]
- Enter a name like **powers** and tap Save



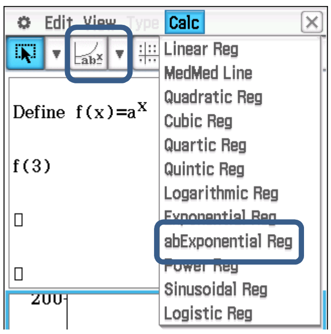
Graph derivative

- Select 📈 from the graph pull-down menu
- Highlight cells C5 to C12
- Tap 📈 to draw the graph



Equation of derivative

- In the graph window select
  [Calc | Regression | abExponential Reg]
  Or select and tap [abx]
- Record the equation

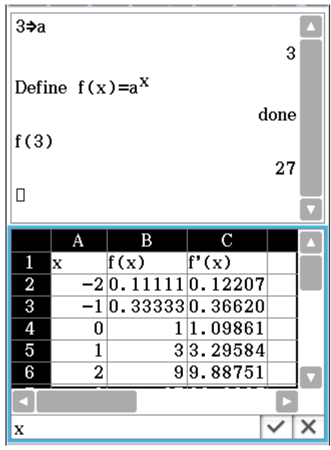


Change the value of a and repeat

- Close the regression window
- Edit the value in Main
- Press EXE

Recalculate equation of gradient function

- Tap in the spreadsheet window
- Select [File | Recalculate]
- Highlight cells C5:C12
- Draw the graph
- Determine equation of the gradient function



Repeat with other values of a aiming to get closer to a function that has derivative values the same. Record your results in the table below.

a	Function $y = a^x$	Equation of derivative (using abExponential regression)
2	$y = 2^x$	$y = 0.6931 \times 2^x$
3		

4. Algebraically

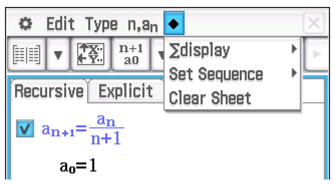
Consider $f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$ where $5! = 5 \times 4 \times 3 \times 2 \times 1$

a) Calculate the first five terms of $f'(x)$.

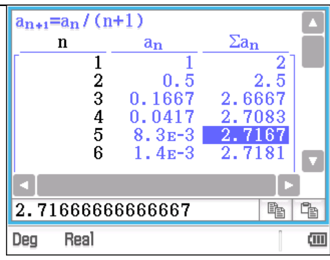
b) Calculate $f(1)$ and $f'(1)$ based on the first five terms using the sequence application.

Sum the terms

- Open Sequence App
- Select [Type | a_{n+1}Type a_0]
- Select [n,a_n | a_n]
- Press ÷
- Select [n a_n | n] then enter + 1
- Set a_0 to 1, as shown



- Tap [icon] to set values to display from 0 to 20
- Select [♦ | Σdisplay] Ensure it is On
- Tap [icon] to display table
- Tap [Resize icon] to see more of the table



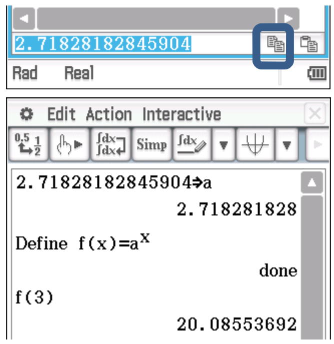
c) After how many terms does the displayed value stop changing? Make sure you check the number as displayed at the bottom of the screen (i.e. 14 d.p.).

d) What is this value?

Copy the value, return to [Main √α] and store this value as a.

Rerun the spreadsheet

- Tap [icon] to copy the number or [Edit | Copy]
- Tap [Main √α]
- Paste the new value and store as a (Use [Edit | Paste]) and press [EXE]
- Open Graph&Table
- Alter function f(x) to a^x
- Open the spreadsheet, recalculate, draw the graph and do the regression



e) Redraw the graphs. What do you notice?

f) Use your spreadsheet and the abExponential regression to determine the equation of the derivative function.

Learning Notes

This actvity is unusual in that so many ClassPad applications are being used with links between them. Being flexible and using the appropriate tool can make your work faster.

Detailed instructions for building the spreadsheet:

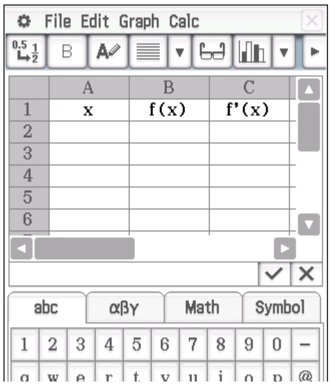
Enter column headings

- Open the Spreadsheet application
- Select [File | New]

Enter headings

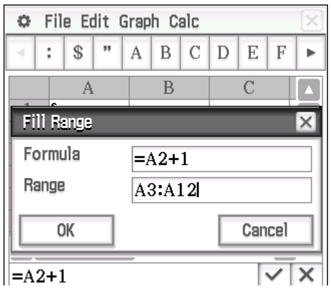
Use the [Keyboard]

- Tap in cell A1, enter **x** and tap ☑ or press [EXP]
- Tap in cell B1, enter **f(x)** and tap ☑
- Tap in cell C1, enter **f ′(x)** and tap ☑



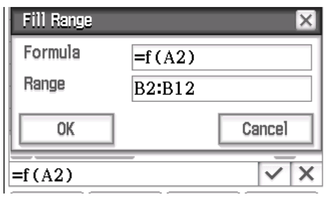
Enter *x*-values

- Tap in cell A2, enter **–2** and tap ☑
- Tap in cell A3, enter **=A2+1** and tap ☑
  starting the entry with = indicates a formula. In this case it is one more than the cell above.
- Tap in cell A3 again
- Select [Edit | Fill | Fill Range] and complete dialogue box as shown.
  The colon (:) and capital letters are available at the top of the screen.



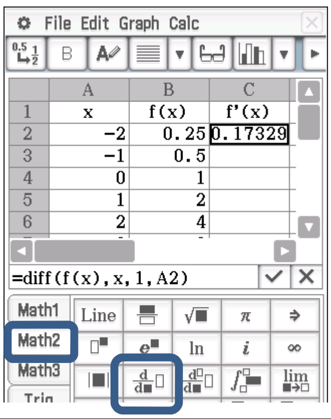
Enter function values

- Tap in cell B2, enter **=f(A2)** and tap ☑
  f(x) was already defined in Main as $f(x) = a^x$.
  This command should calculate $f(-2)$ as –2 is in cell A2.
- Tap in cell B2 again
- Select [Edit | Fill | Fill Range]
- Fill the Range B2:B125



Enter derivative or gradient function values

- Tap in cell C2, enter =diff(f(x),x,1,A2) and tap ☑
  Use the soft keyboard [abc] to enter text and [Math2] for the derivative
- Tap in cell C2 again
- Select [Edit | Fill | Fill Range]
- Fill the Range C2:C125



In Q4 the factorial symbol is used. $n! = n(n-1)(n-2)...1$ i.e. it is defined for positive integers and is the product of each integer between n and 1 inclusive.

The question requires differentiating expressions of the form $\dfrac{x^n}{n!}$.

For example,

$$\frac{d}{dx}\left(\frac{x^7}{7!}\right) = 7\frac{x^6}{7!}$$

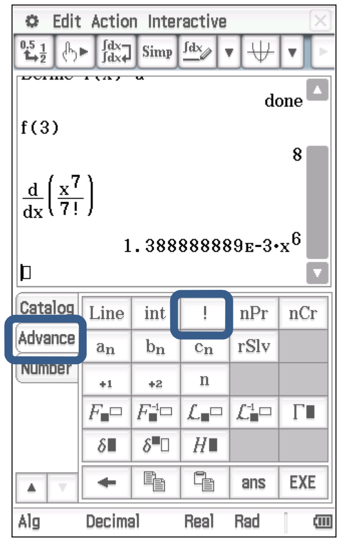
$$= \frac{7x^6}{7\times6\times5\times4\times3\times2\times1}$$

$$= \frac{x^6}{6\times5\times4\times3\times2\times1}$$

$$= \frac{x^6}{6!}$$

Using CAS

- Press Keyboard Math2 tab for $\frac{d}{d\square}\square$ template
- ! is available in the Advance menu



As a result of this activity you should know that

- there is a function with itself as derivative i.e. $\dfrac{dy}{dx} = y$.
- this is an exponential function
- $\dfrac{d}{dx}e^x = e^x$
- $e = 1 + \dfrac{1}{1} + \dfrac{1}{2!} + \dfrac{1}{3!} + \dfrac{1}{4!} + \dfrac{1}{5!} + ...$
- $e \approx 2.718281$

The number e is sometimes called Euler's number after the Swiss mathematician Leonhard Euler.
The number e is of eminent importance in mathematics, alongside 0, 1, π and i. Like the constant π, e is irrational and it is transcendental. The numerical value of e truncated to 50 decimal places is
2.71828182845904523536028747135266249775724709369995

Activity 2 Differentiating exponential functions

Aim: Differentiate a^x from first principles and establish an important limit.

You may have been exposed to the first principles definition of differentiation in previous work.

In general, for a function $y = f(x)$, the derivative $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Consider the function $f(x) = a^x$.

1. For each step of the working below give a brief description or justification.

Working	Justification
$f'(x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$	
$= \lim_{h \to 0} \frac{a^x . a^h - a^x}{h}$	
$= a^x \lim_{h \to 0} \frac{a^h - 1}{h}$	

The working in Q1 establishes the fact that $\frac{d}{dx}\left(a^x\right) = ka^x$, i.e. the derivative is a scalar multiple of the function itself. The multiplier, *k*, is the value of the limit $\lim_{h \to 0} \frac{a^h - 1}{h}$. The next step is to investigate the value of this limit for different values of *a*.

2. Consider the limit: $\lim_{h \to 0} \frac{a^h - 1}{h}$

   a) Explain why the expression $\frac{a^h - 1}{h}$ is undefined **at** $h = 0$ regardless of the value of *a*.

b) By evaluating the expression at values of h **very close to zero**, an estimation of the value of the limit can be established. Create the spreadsheet below to investigate the value of the limit for different values of a.

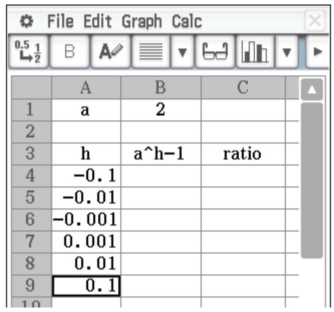
Create the spreadsheet

- Open Spreadsheet application
- Duplicate the headings and Column A as shown

Formulae

- In cell B4 enter the formula =$B$1^A4 – 1
  Fill the formula down to B9
- In cell C4 enter the formula =B4/A4
  Fill the formula down to C9

Detailed instructions are included in the Learning notes



Change the value of a and hence estimate the value of the limit $\lim_{h \to 0} \frac{a^h - 1}{h}$ for the given values of a.

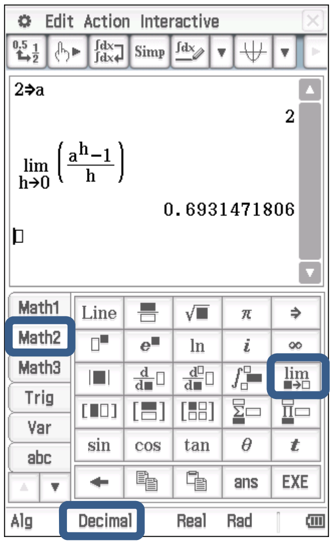
a	Estimate of $\lim_{h \to 0} \frac{a^h - 1}{h}$
2	
3	
4	
2.6	
2.7	
2.8	
2.71828	

c) Explain the significance of the last row in the table above given the first principles derivative $\frac{d}{dx}\left(a^x\right) = a^x \lim_{h \to 0} \frac{a^h - 1}{h}$.

3. Use CAS to confirm limits

Evaluate limits

- Open Main
- Ensure calculator is in Decimal mode
- Store a value for a
- Press [Keyboard]
- Tap [Math2]
- Tap [lim]
- Complete the limit calculation



a) Evaluate the limits below by changing the value stored as a.

(i) $\lim_{h\to 0}\frac{3^h-1}{h}$

(ii) $\lim_{h\to 0}\frac{2.7^h-1}{h}$

(iii) $\lim_{h\to 0}\frac{2.718^h-1}{h}$

b) Solve the equation for a: $\lim_{h\to 0}\frac{a^h-1}{h}=1$

(i) With calculator in Decimal mode

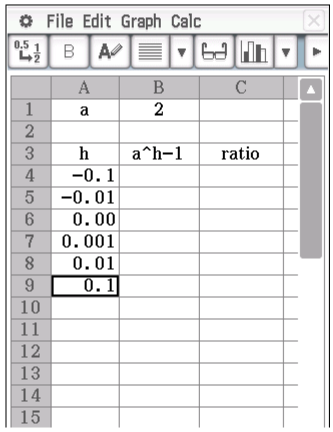
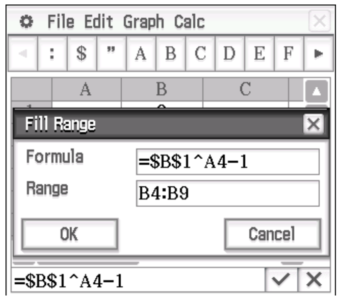
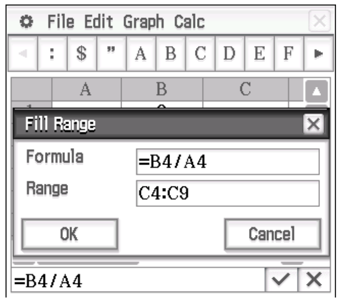
(ii) With calculator in Standard mode

c) Hence solve the equation for a: $\frac{d}{dx}\left(a^x\right)=a^x$

Learning Notes

In the previous activity you explored the idea of a function that is its own derivative. This activity looks more formally at this concept by differentiating the exponential function $f(x) = a^x$ from first principles. This leads to the limit $\lim_{h \to 0} \frac{a^h - 1}{h}$. The concept of a limit may not be familiar to you. Essentially we are looking at the behaviour of an expression close to a particular value, in this case when h is close to zero. The concept of a limit will be developed further in the activity *Looking at limits*.

Q2 Detailed instructions for the creation of the spreadsheet

Enter column headings • Open the Spreadsheet application • Select [File	New] Enter headings Use the Keyboard • Tap in cell A1, enter **a** and tap ☑ • Tap in cell B1, enter **2** and tap ☑ • Tap in cell A3, enter **h** and tap ☑ • Tap in cell B3, enter **a^h–1** and tap ☑ • Tap in cell C3, enter **ratio** and tap ☑ • Tap in cell A4, enter **–0.1** and tap ☑ • Repeat for other values of h in column A	
Calculate a^h-1 • Tap in cell B4, enter **=$B$1^A4–1** and tap ☑ • Tap in cell B4, then [Edit	Fill	Fill Range] and complete dialogue box as shown. Note the colon (:), $ and capital letters are available at the top of the screen	
Calculate ratio • Tap in cell C4, enter **=B4/A4** and tap ☑ • Tap in cell C4 • Select [Edit	Fill	Fill Range] • Fill the Range C4:C9 as shown	

The exact derivative of a^x, $\frac{d}{dx}\left(a^x\right) = \ln(a) \cdot a^x$, involves logarithms. These will be covered in Chapter 4 at which point it should become clear why $a = e$ is the only solution to the equation in Q3c).

Activity 3 Route 2.7…*e*

Aim: Investigate continuously compounding interest, define *e* as a limit and connect to continuous growth.

Compound interest

When money is invested in a bank account, interest is paid by the bank, most commonly using compound interest – that is, interest is added to the account which increases the balance. In this way, interest is calculated on an increasing balance over time. The frequency with which the compound interest is calculated varies.

Consider the following situation:
A \$10 000 lump sum is invested in an account paying 10% per annum (p.a.) compound interest for one year.

1. Determine the value of the investment after one year assuming interest is compounded yearly.

2. Determine the value of the investment after one year assuming interest is compounded biannually.
   Note that the 10% p.a. becomes 5% for each six month period.

Investigate what happens when interest is compounded more frequently.
You may like to use the Financial application (see Learning notes) or the formula

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

Where,
A = amount in account at the end of the time period
P = principal (initial amount invested)
r = interest rate per annum, as a decimal
n = number of times interest is compounded per year
t = number of years over which money is invested.

3. Complete the table

	Compounding period	Amount after one year
a)	quarterly	
b)	monthly	
c)	weekly	
d)	daily	
e)	hourly	

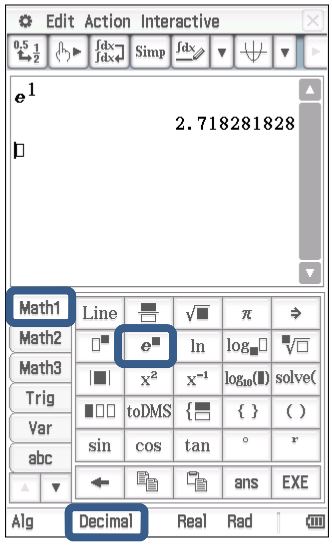
4. Consider the situation if interest rates were 100% per annum.
Complete the table for an initial investment of \$1.
(No rounding, apart from the calculator limits.)

	Compounding period	Amount after one year
a)	yearly	
b)	biannually	
c)	quarterly	
d)	monthly	
e)	weekly	
f)	daily	
g)	hourly	

5. You will notice that the value of the investment with increasingly frequent compounding is approaching an upper limit. In Q4 the limit is *e*, Euler's number. Record the value of *e* to 9 decimal places.

Enter *e*

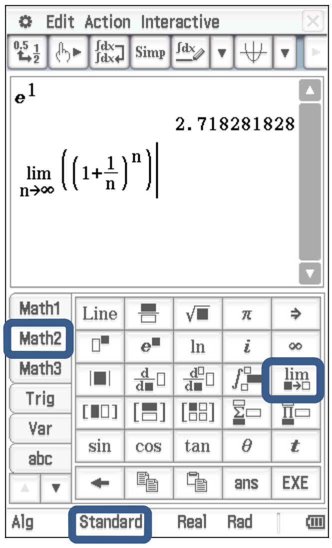
- Open Main
- Check ClassPad is in Decimal mode
- Press Keyboard
- tap $e^{\blacksquare}$ the exponential template in the Math1 menu
- Enter 1 for the exponent
- Press EXE



6. Evaluate the following limits:

Evaluate limits

- Open Main
- Change from Decimal to Standard mode
- Press Keyboard
- Tap Math2
- Tap lim
- Complete the entry for desired limit calculation



a) $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n$

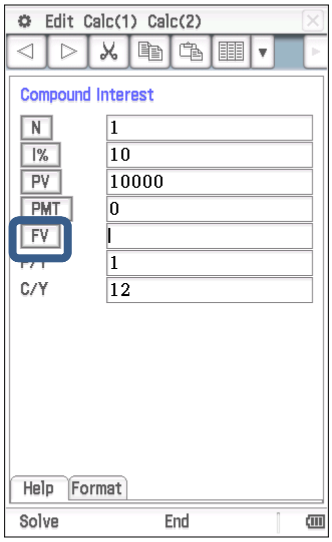
b) $\lim\limits_{n\to\infty}\left(1+\dfrac{a}{n}\right)^n$

Learning notes

Euler's number, *e*, was present in our compound interest example when the number of compounds per year became **infinite**. This represents **continuous growth** as compared to a number of discrete increases over time. In fact, whenever a quantity undergoes continuous exponential growth (or decay) Euler's number will be involved.

Use Financial app

- Tap Menu
- Open Financial app
- Select Compound Interest
- tap Help, it is useful to show the Help
  - The screen shot is set up for 1 year
  - 10% p.a. interest
  - An initial investment of $10 000
  - $0 repayment
  - And interest calculated 12 times per year
- Tap FV to calculate the final amount (You can ignore the negative sign)
- Change C/Y and tap FV for a different compounding period



Q6 Have your calculator in Standard mode to get exact answers.
You may want to [Edit | Clear All Variables] to ensure a is not defined from the previous activity.

Activity 4 Growth and decay

Aim: Model growth and decay situations using exponential functions.

> **Free jabs to fight disease outbreak**
>
> The State Government will fund a free whooping cough vaccine for parents of newborn babies in a bid to control the biggest outbreak of the potentially fatal infection since 2004.
>
> Excerpt from The West Australian 18/01/2011

Pertussis (commonly known as whooping cough) is an airborne, highly contagious bacterial disease. In Western Australia, the Department of Health record and publish data on suspected cases of communicable diseases, including Pertussis. The recorded cases for the state over a number of years is shown in the table below.
Source:
http://www.health.wa.gov.au/diseasewatch/vol16_issue3/review_of_notifiable_diseases_2011_table1.cfm

Year	Pertussis notifications in WA by year
2007	134
2008	467
2009	786
2010	1458
2011	4021

1. Assume $t = 0$ corresponds to 2007, $t = 1$ corresponds to 2008 and so on. Enter the data into the Statistics application and determine an exponential equation of best fit of the form $P = P_0e^{kt}$.

2. Differentiate the equation from Q1 and show that it satisfies the differential equation $\frac{dP}{dt} = kP$.

3. Explain the meaning of the k value in terms of the rate of growth of the number of notifications of Pertussis.

4. Use your model to predict the number of notifications of Pertussis in WA in 2012.

5. In which year is the number of notifications expected to first exceed 20 000?

6. Explain the limitations of the exponential model for predicting future values.

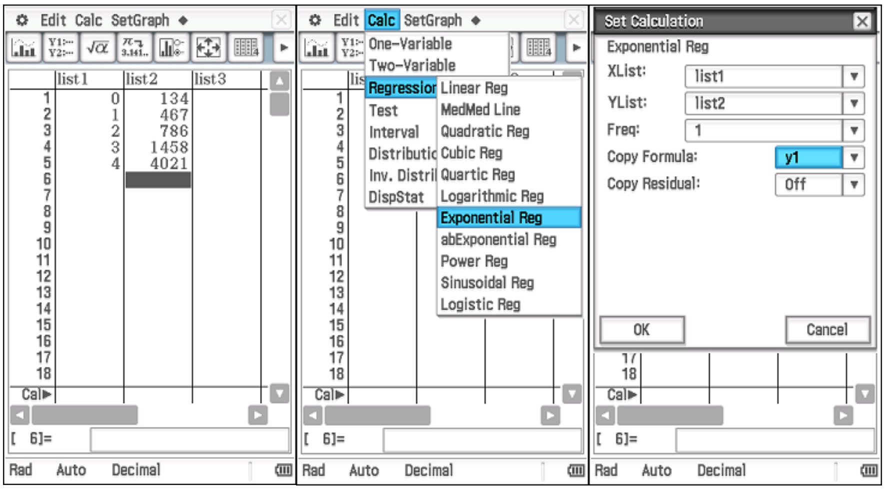
Half-life

Radioactive decay is the loss of energy by an unstable atomic nucleus. The number of unstable nuclei of a radioactive substance remaining at some point in time can be modelled using an exponential function of the form $A = A_0 e^{\lambda t}$ where λ is the decay constant and $\lambda < 0$, and A_0 is the initial amount present. The half-life is the amount of time required for half of an amount of a radioactive substance to decay.

7. A radioactive substance with a half-life of 60.5 days has been decaying for 10 days. Currently 150g of the substance remains. Determine how much of the substance was present initially.

Learning notes

Q1 The screen shots below indicate the process of obtaining the equation of best fit.



"Copy Formula" is optional, but it inserts the equation into the Graph and Table app which is more functional than working with the graph in the Statistics app. It also allows the use of the formula in the Main screen which can be useful for future value predictions. Simply type y1(x) to recall the function.

Q3 The differential equation $\frac{dP}{dt} = kP$ indicates that the population's **instantaneous rate of change is proportional to the size of the population** at that instant. Real life examples of this type of growth/decay are numerous and you are encouraged to research further.

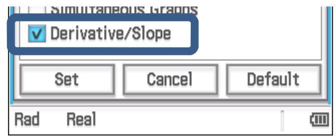
Q7 Hint: Use the half-life information to determine the decay constant λ before proceeding.

Activity 5 Differentiating trigonometric functions

Aim: Investigate the derivative of the sine and cosine functions.

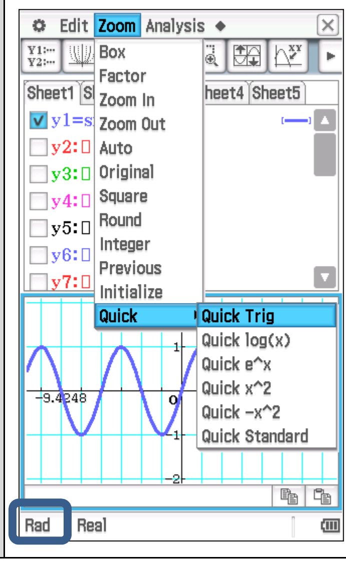
Set up

- Select [⚙ | Graph Format] and ensure Derivative/Slope is checked
- Tap Set



Graph y=sin(x)

- Open Graph&Table app
- Ensure your calculator is in radian mode
- Enter the function
- Tap [graph icon] to draw the graph
- Select [Zoom | Quick | Quick Trig] for a nice view window



1. Consider the graph of $y = \sin(x)$.

   a) Explain why the gradient function graph should be periodic.

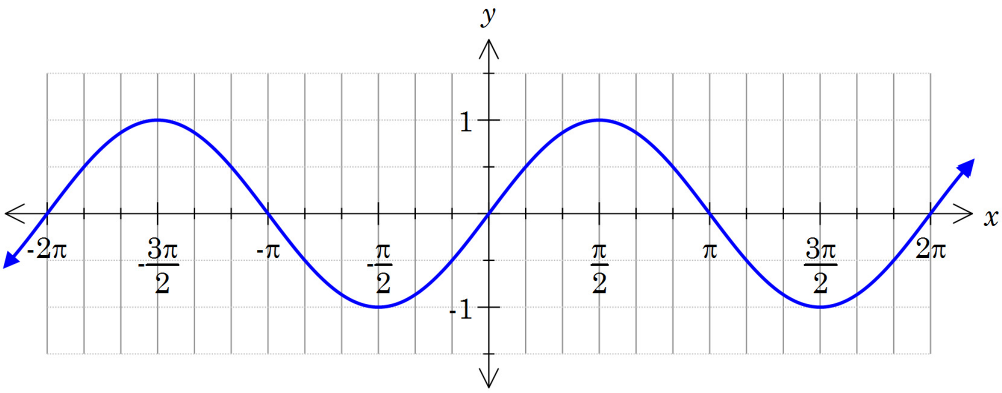
   b) What would be the period of the graph of the gradient function?

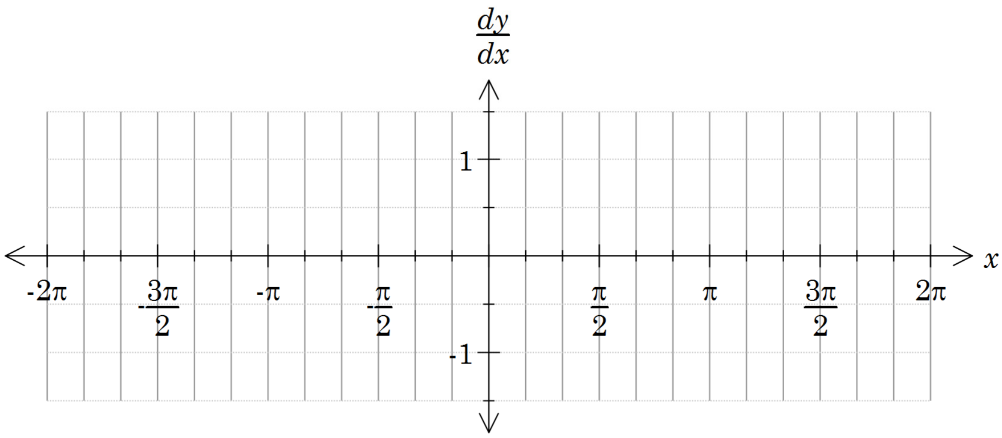
   c) Where on the graph of $y = \sin(x)$ is the gradient zero?

   d) Sketch a picture of what the gradient function for $y = \sin(x)$ should look like in the space below.

2. The graph of $y = \sin(x)$ is shown below.

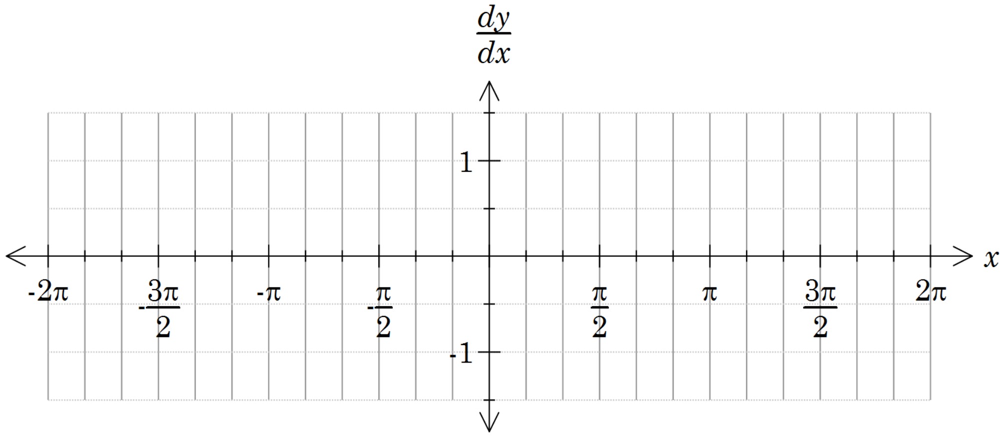
a) Use the Trace feature to determine the gradient of the curve at various points along the curve. Plot these on the derivative axes below and hence sketch a curve to represent the derivative of $y = \sin(x)$.





b) Suggest a function for the derivative graph.

3. Repeat Q2 for the graph of $y = \cos(x)$.



We can use the first principles definition of a derivative, $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ to differentiate $y = \sin(x)$.

4. For each step of the working below give a brief description or justification.

Working	Justification
$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$	
$= \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$	
$= \lim_{h \to 0} \frac{\sin(x)\cos(h) - \sin(x)}{h} + \lim_{h \to 0} \frac{\cos(x)\sin(h)}{h}$	
$= \sin(x) \lim_{h \to 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h \to 0} \frac{\sin(h)}{h}$	
$=$	

To proceed, we need to evaluate the two limits. This would be a sensible time to complete the two activities *Looking at limits* and *Sine of x on x.*

Alternatively, you can evaluate the limits using the limit template in Main, then complete the final line of working.
See Learning Notes for details.

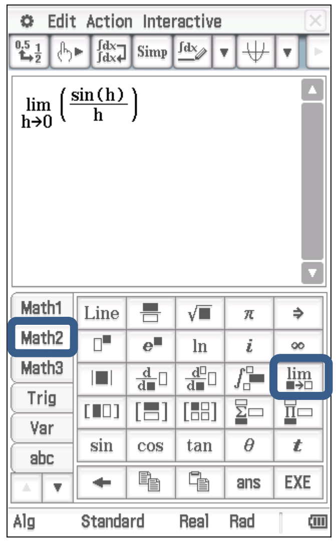
5. Differentiate $y = \cos(x)$ from first principles.

Learning notes

Q4 Whilst the limits can be evaluated using CAS, you are encouraged to complete the two activities *Looking at limits* and *Sine of x on x* to appreciate the numerical and geometrical arguments that help with the understanding behind them.

Evaluate limits

- Open Main
- Press Keyboard
- Tap Math2
- Tap lim
- Complete the limit calculation



Activity 6 The second derivative

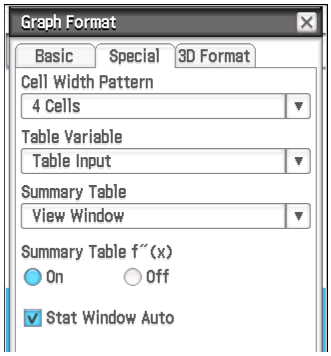
Aim: Use CAS and the second derivative test to determine the nature of stationary points.

1. Consider the graph of $y = f(x)$ where $f(x) = \frac{x^5}{5} - x^4 + \frac{2x^3}{3} + 2x^2 - 3x + 1$.

   Use ClassPad to display a table of values for y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $-3 \le x \le 4$.

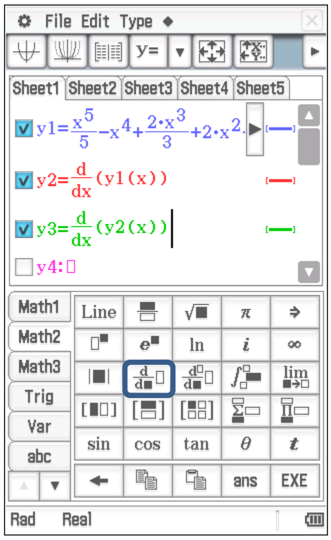
Set up

- Tap ⚙ select Graph Format
- Untick Derivative/Slope
- Tap Special tab
- Set Cell Width Pattern to 4 cells



Define the functions

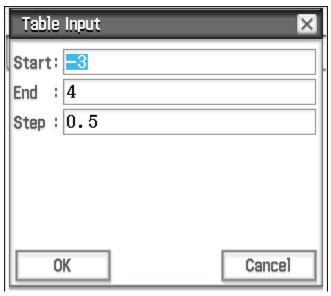
- Set y1 to $\frac{x^5}{5} - x^4 + \frac{2x^3}{3} + 2x^2 - 3x + 1$ i.e. $f(x)$
- Set y2 to $f'(x)$ and y3 to $f''(x)$
- Open the Keyboard Math2 tab
- Tap $\frac{d}{dx}\square$ for the derivative template
- Tap abc to enter y and x
- ⮌ to return to other keyboard menus



Set table input

- Tap ⊞ to access the Table Input window. Set values as shown and tap OK
- Tap ▦ to display the table

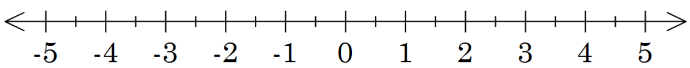
  You may like to switch to full screen (tap Resize) to make it easier to see the values



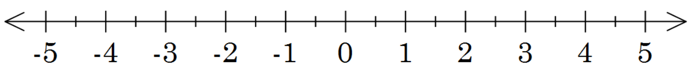
a) Determine approximate x-values for when either $f(x)=0$, $f'(x)=0$ or $f''(x)=0$ and record these x-values in the table below. Also calculate and record approximate values for the function, the derivative and the second derivative at these points.

x	$f(x)$	$f'(x)$	$f''(x)$

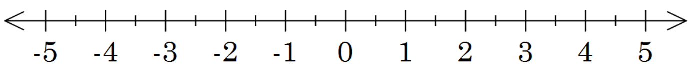
b) Draw a sign diagram for $f(x)$



c) Draw a sign diagram for $f'(x)$

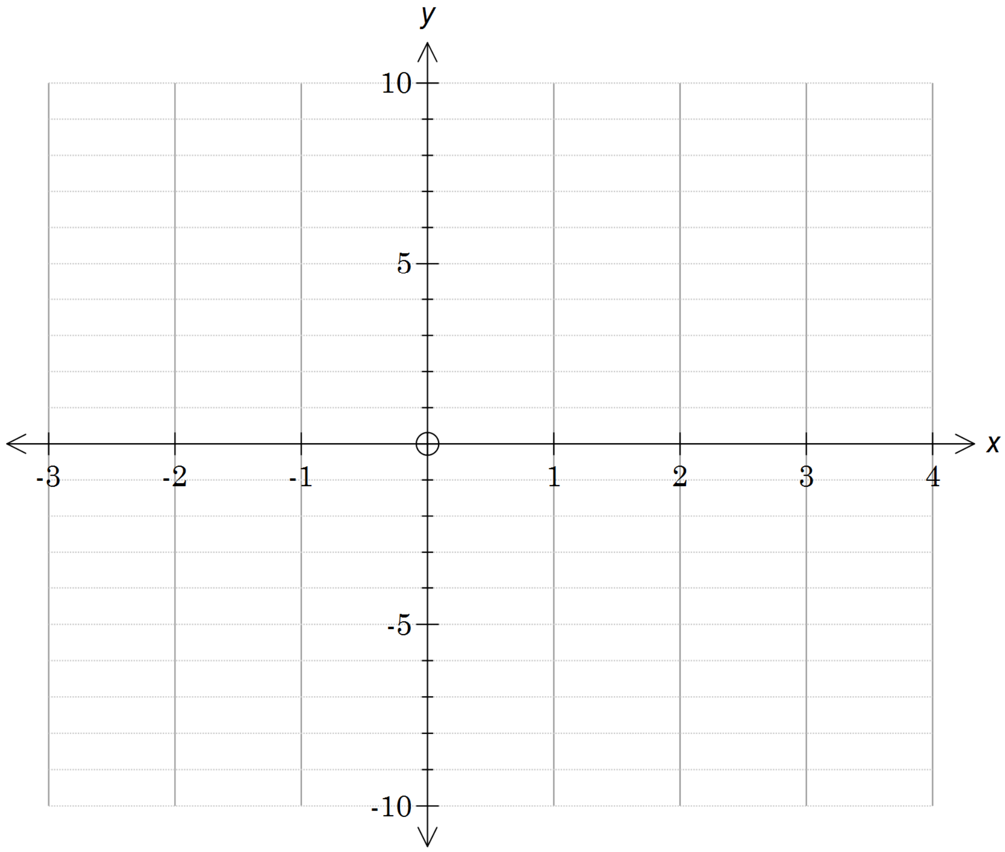


d) Draw a sign diagram for $f''(x)$



e) Sketch the graph of $y = f(x)$.

Show the x-intercepts and stationary points and any points of inflection.



Note: Plotting key features first is useful for transcribing graphs from technology to paper.

2. Choose one of each alternative function/gradient and increasing/decreasing/indeterminate to make true statements. Put lines through the inappropriate words.

For the function $y = f(x)$, if:

a) $f'(x) > 0$ the function/gradient is increasing/decreasing/indeterminate.

b) $f'(x) = 0$ the function/gradient is increasing/decreasing/indeterminate.

c) $f'(x) < 0$ the function/gradient is increasing/decreasing/indeterminate.

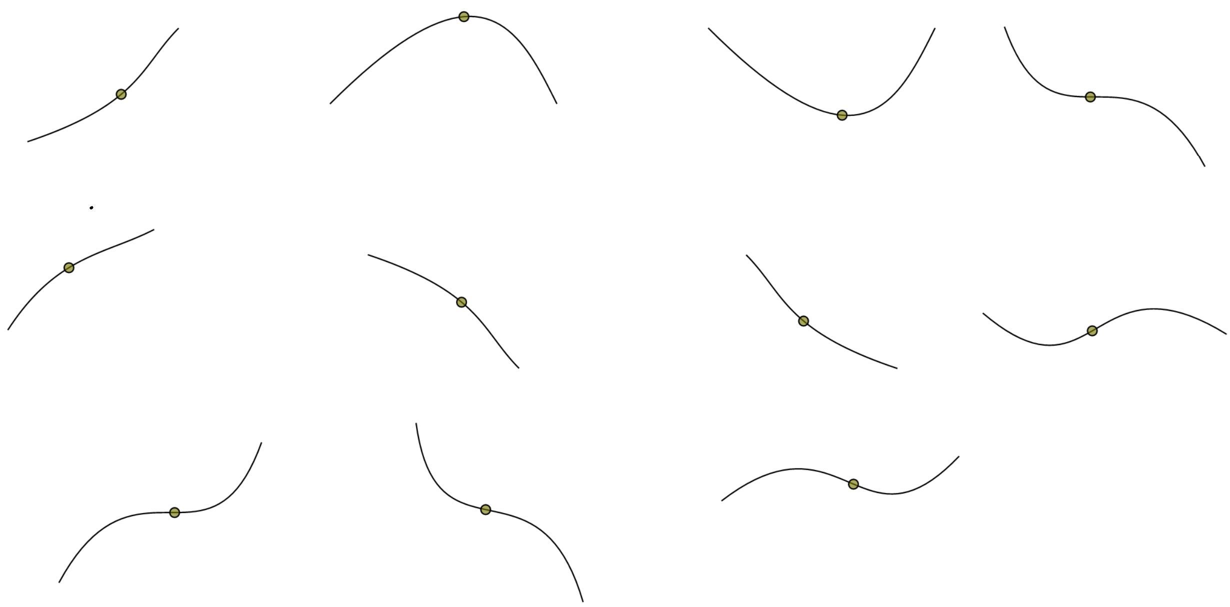
d) $f''(x) > 0$ the function/gradient is increasing/decreasing/indeterminate.

e) $f''(x) < 0$ the function/gradient is increasing/decreasing/ indeterminate.

f) $f''(x) = 0$ the function/gradient is increasing/decreasing/ indeterminate.

3. In each cell of the table draw all curve segments (drawn below) that could fit the given conditions.

	$f''(x) > 0$	$f''(x) < 0$	$f''(x) = 0$
$f'(x) > 0$			
$f'(x) = 0$			
$f'(x) < 0$			



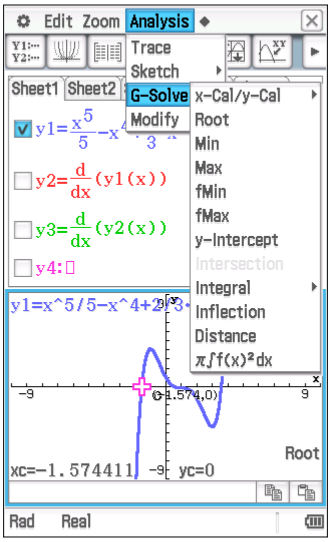
Learning notes

Q1 a) You can use approximate values according to your table. Look for changes in sign (from positive to negative or vice versa) in a column that will indicate a zero has occurred. If you want more precise values, you could use CAS to solve (for example $y1(x) = 0$ for roots).

Sign diagrams indicate whether the quantity is positive or negative. The quantity, for example a function may change at an intercept or vertical asymptote, i.e. where the quantity is 0 or undefined.

To draw a sign diagram, first determine the critical values, points where the quantity is 0 or undefined. Then consider whether the quantity is positive or negative in between these values.

Another way of thinking about sign diagrams is to look at the graph and if it is above the axis put a +, if below put a –.
To locate non-integral critical values more accurately (not required for this activity) use [Analysis | G-Solve | Root]. It is also easier if just one function is drawn at the time.



Q2. For this question use the definition of a strictly increasing function: a function where the gradient is greater than 0 and a strictly decreasing function is one where the gradient is less than 0. For many the term increasing function means the gradient is not negative.
Interpret indeterminate as neither strictly increasing, nor strictly decreasing or unknown.

Q3. Use your answer to Q2 to help.
Note: when both the first and second derivatives are zero at a point we don't know whether the gradient is increasing or decreasing. Use a sign diagram of $f'(x)$ to determine the nature of such a stationary point.

Activity 7 Graphing functions

Aim: Use calculus to determine key features of graphs.

When drawing graphs of functions, we plot points of interest. While ClassPad will draw the function for us, in this section of the course it is expected that you understand how calculus is used to determine stationary points (points where the gradient is 0) and their nature.

1. Consider the graph of $y = f(x)$ where $f(x) = x - \frac{4}{3} + \frac{4}{9x}$.

   a) Duplicate the CAS working on your ClassPad to determine the stationary points of the function and their nature.

	By-hand solution	CAS working
(i)	$y = x + \frac{4}{9x} - \frac{4}{3}$ $\frac{dy}{dx} = 1 - \frac{4}{9x^2}$	Edit Action Interactive Define $f(x)=x-\frac{4}{3}+\frac{4}{9x}$ done $\frac{d}{dx}(f(x))$ $\frac{9\cdot x^2-4}{9\cdot x^2}$
(ii)	$\frac{dy}{dx} = 0$ $\Rightarrow 1 - \frac{4}{9x^2} = 0$ $\Rightarrow \frac{9x^2 - 4}{9x^2} = 0$ $\Rightarrow \frac{(3x+2)(3x-2)}{9x^2} = 0$ $\Rightarrow x = \pm\frac{2}{3}$	$\frac{9\cdot x^2-4}{9\cdot x^2}$ solve(ans=0 $\left\{x=-\frac{2}{3}, x=\frac{2}{3}\right\}$
(iii)	$\frac{d^2y}{dx^2} = \frac{8}{9x^3}$ $\left.\frac{d^2y}{dx^2}\right\vert_{x=\frac{-2}{3}} = -3 < 0 \Rightarrow x = \frac{-2}{3}$ is a local maximum $\left.\frac{d^2y}{dx^2}\right\vert_{x=\frac{2}{3}} = 3 > 0 \Rightarrow x = \frac{2}{3}$ is a local minimum	$\frac{d^2}{dx^2}(f(x))$ $\frac{8}{9\cdot x^3}$ $\frac{d^2}{dx^2}(f(x))\vert x=-\frac{2}{3}$ -3 $\frac{d^2}{dx^2}(f(x))\vert x=\frac{2}{3}$ 3

(iv)	$y\big\|_{x=\frac{2}{3}} = 0$ $\therefore$ local minimum at $\left(\frac{2}{3}, 0\right)$ $y\big\|_{x=\frac{-2}{3}} = \frac{-8}{3}$ $\therefore$ local maximum at $\left(\frac{-2}{3}, \frac{-8}{3}\right)$	$f\left(\frac{2}{3}\right)$ 0 $f\left(\frac{-2}{3}\right)$ $-\frac{8}{3}$

b) For each of the steps (i) to (iv) describe the reason for that working.

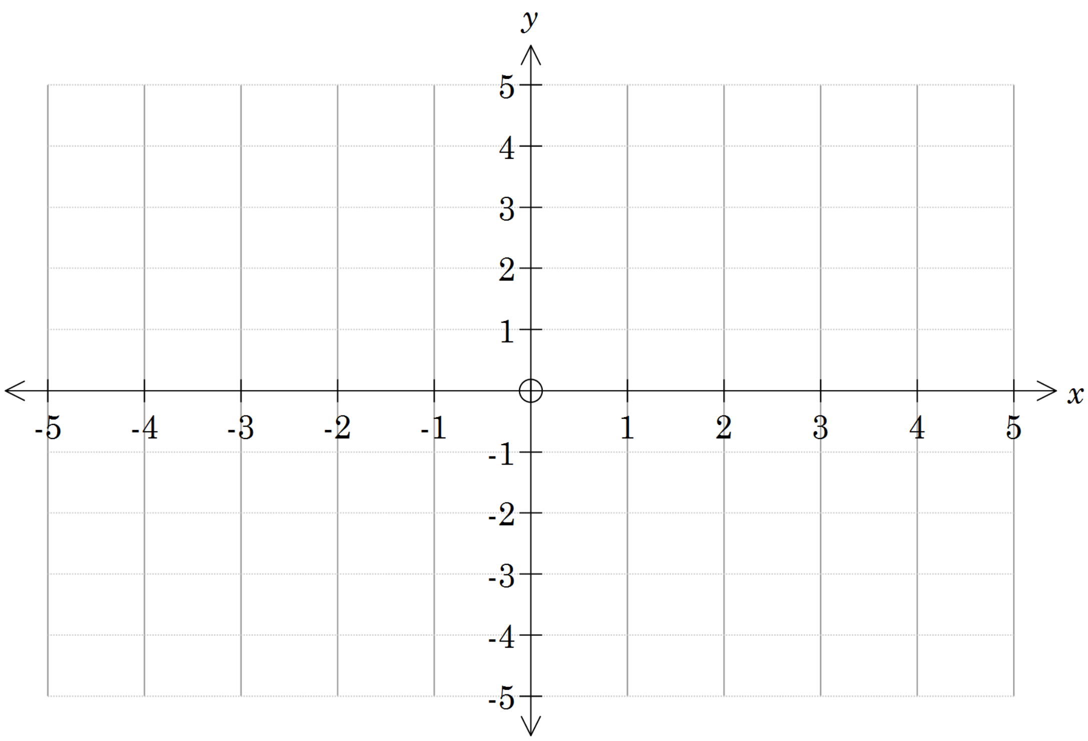
(i)

(ii)

(iii)

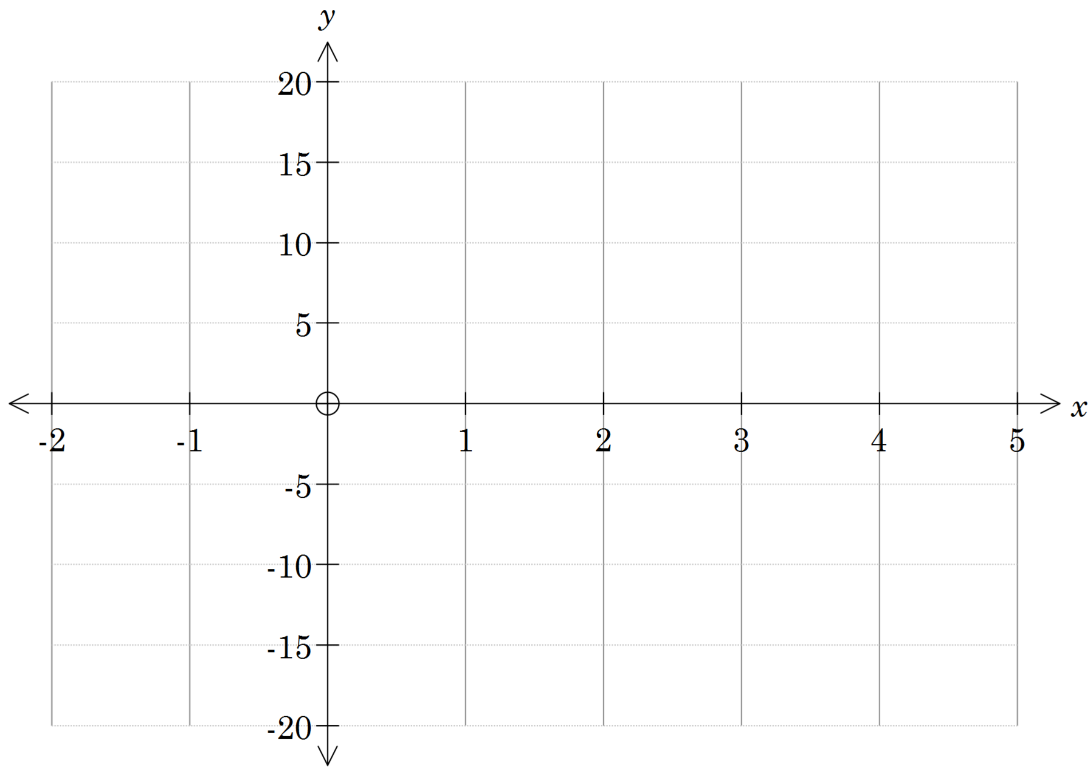
(iv)

c) Draw the graph of $y = f(x)$ showing key features.
(e.g. stationary points, intercepts and asymptotes)

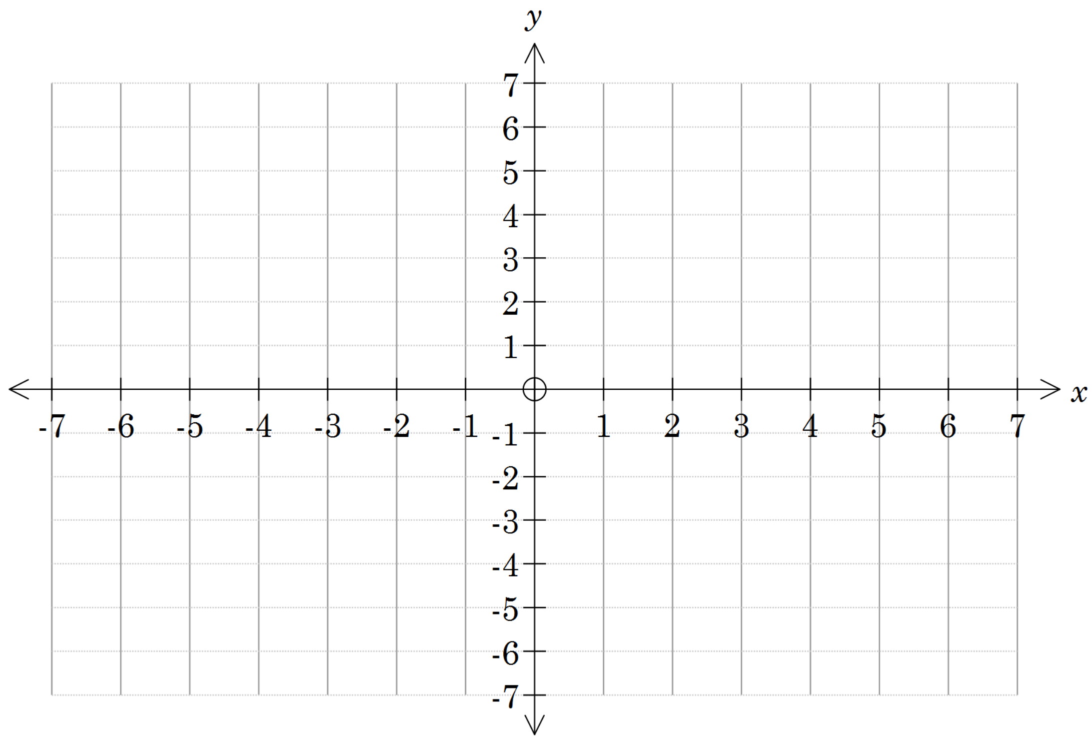


2. Draw the graph of each function on the grid provided. Calculate, plot and label key features of the graphs.

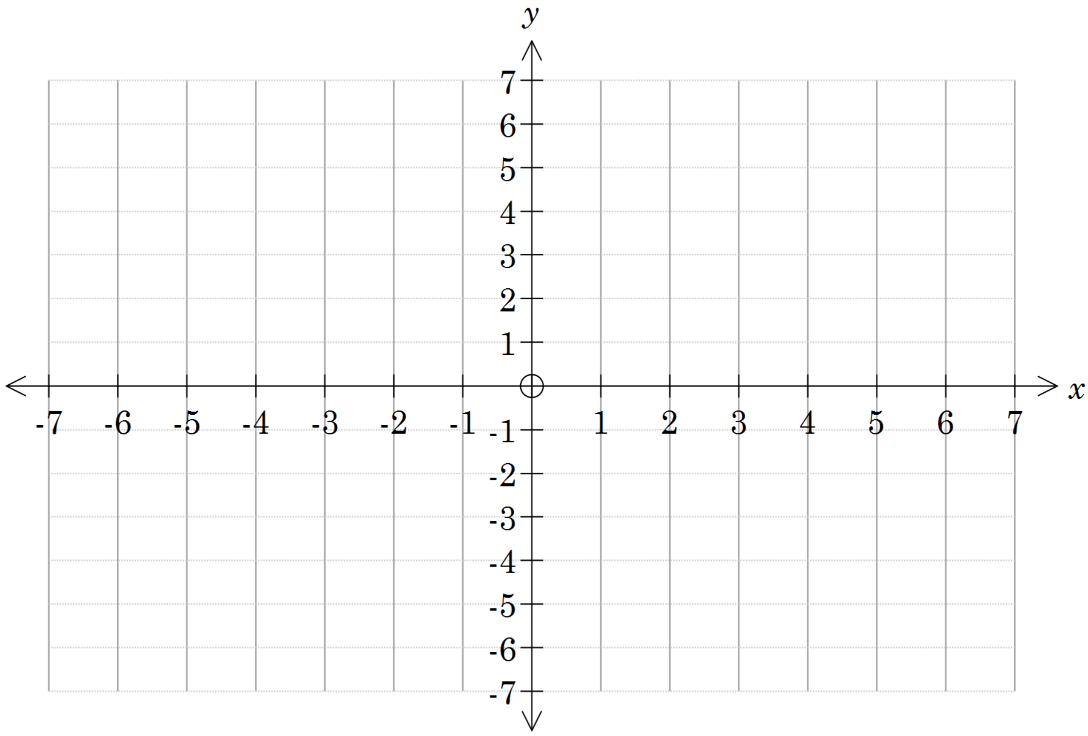
a) $y = x^4 - 6x^3 + 9x^2 + 4x - 13$



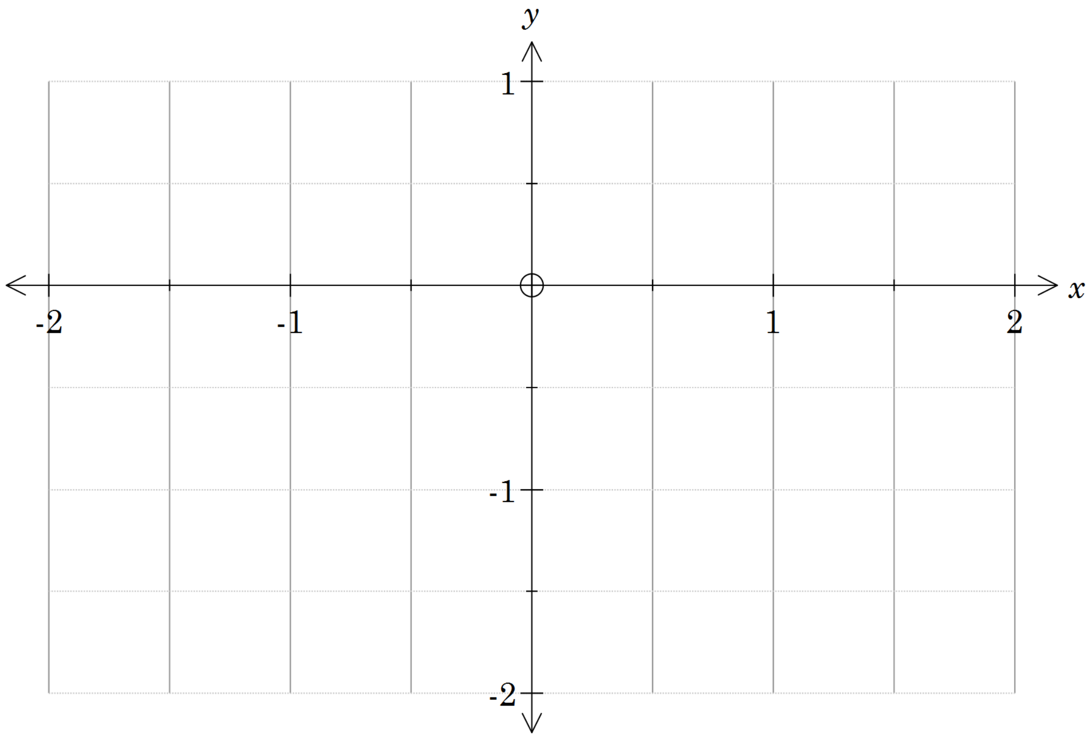
b) $y = x\sin x, \ -2\pi \leq x \leq 2\pi$



c) $y = \dfrac{x^3 - 9}{x^2 + 2}$



d) $y = x^6 - x^4 - 1$



Learning notes

The intention in this activity is to use CAS to determine the key features. Being able to state why you are doing things is valuable in terms of the processes you want to do when working without the calculator. Using the calculator means that we can handle a much greater range of functions. The idea that we need to solve an equation is the focus rather than the mechanics of the calculation.

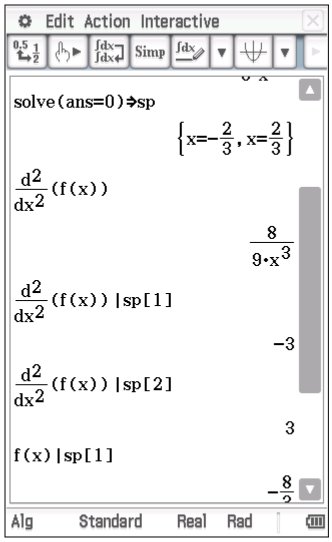
Use of CAS

An interesting modification to the CAS working in Q1 a) is to store the result for $f'(x)=0$ as a variable.

The screen shot shows the result being stored as variable sp.

To evaluate the second derivative at points where the gradient is 0 instead of retyping or dragging the answer in we can now refer to the first element of sp as sp[1] i.e. $x=-\dfrac{2}{3}$.

Now by altering the function definition the working will recalculate and provide us with the first two stationary points.



Q1 b) In describing the steps try to summarise with a phrase or sentence, e.g. Stationary points occur when $f'(x)=0$.

When sketching a graph that you have displayed using technology:

- Ensure the window is appropriate, i.e. match the calculator window to the grid provided or adjust the scale to show the features you want.
- Calculate values for the key features.
- Plot the key features.
- Sketch the graph.

Key features of graphs will vary, depending on the function. You may wish to include:

- intercepts
- stationary points (local maxima, minima and horizontal points of inflection)
- non-horizontal points of inflection
- asymptotes and behaviour as $x\to\pm\infty$.

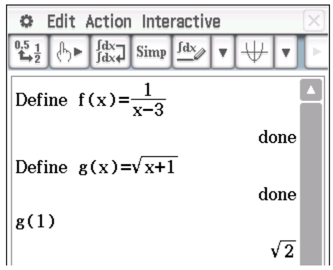
Activity 8 Composite functions

Aim: Investigate composition of functions and associated domains and ranges.

Consider the functions $f(x) = \dfrac{1}{x-3}$ and $g(x) = \sqrt{x+1}$.

Define functions

- In Main $\sqrt{\alpha}$, select [Interactive | Define] to store the functions
- The functions can then be called upon using the abc keyboard or by copying and pasting and changing the input



1. a) Use CAS to evaluate the following:

   (i) $g(3)$

   (ii) $f(2)$

   (iii) $f(g(3))$

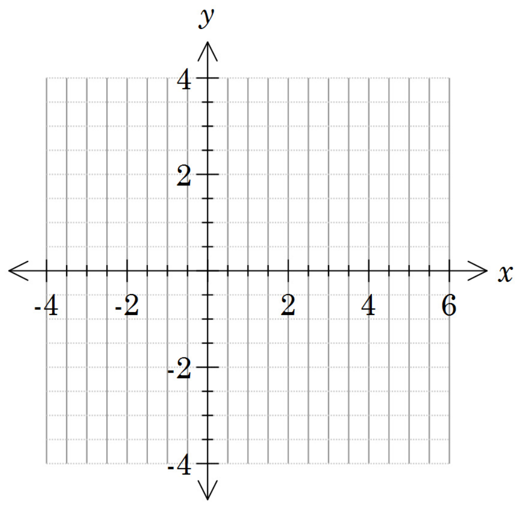
   b) Explain the significance of your answers in (ii) and (iii) above.

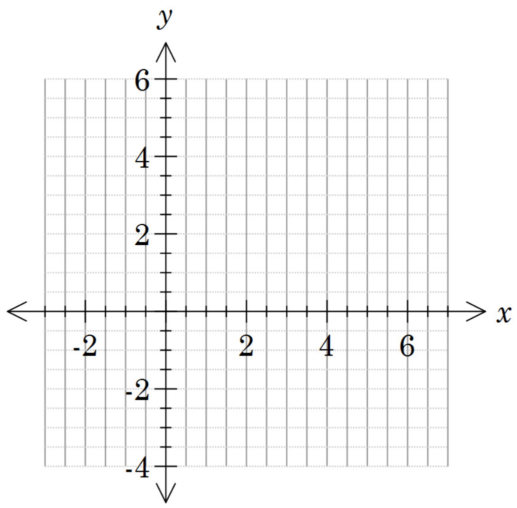
2. Use CAS to evaluate $f(g(x))$.

3. a) Use CAS to evaluate $g(f(x))$.

   b) Is the answer to Q3a) as you expected? Manually substitute $f(x)$ into $g(x)$ and manipulate to show it is equivalent to your answer to a).

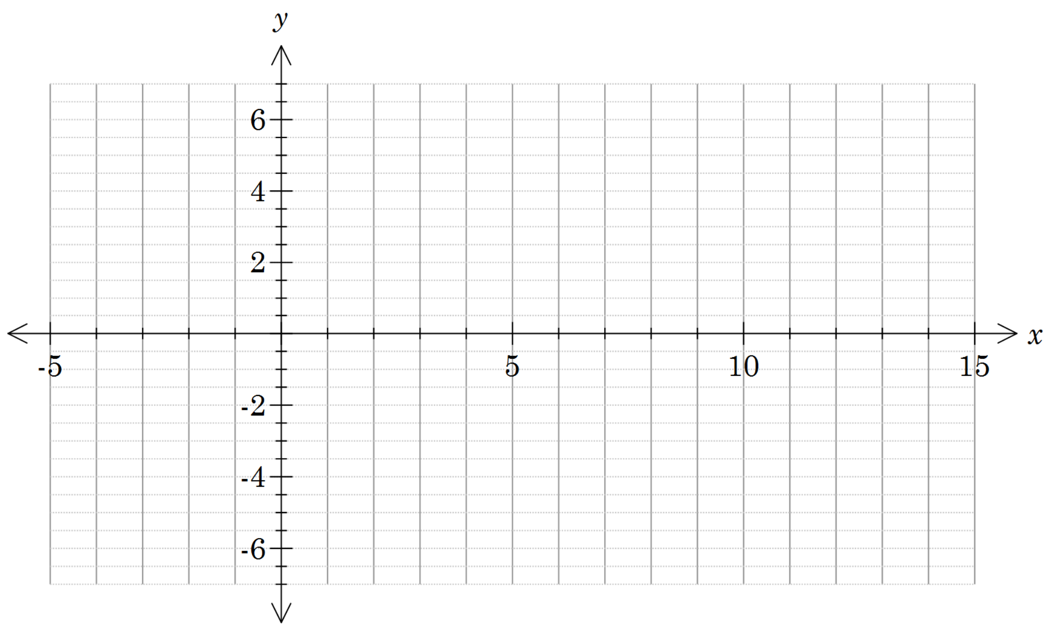
4. a) Graph each of the functions $y = f(x)$ and $y = g(x)$ on your ClassPad, and sketch the graphs on the axes below.





b) State the domain and range of each function.

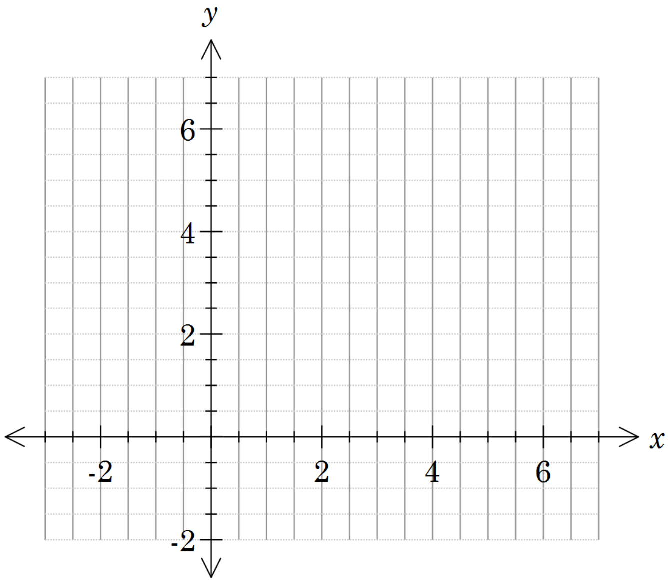
5. Display the graph of $y = f(g(x))$ and sketch on the axes below. State its domain and range.



6. a) Display the graph of $y = g(f(x))$ on your calculator.

b) Investigate the behaviour of the composite function when approaching $x = 3$ from the right. (Note: The calculator display may be deceiving! Use [Analysis | Trace] and type some values slightly larger than 3 to investigate.)

c) Draw a neat sketch of the graph of $y = g(f(x))$. State the associated domain and range.



d) Display the graph of $y = \dfrac{1}{x-3} + 1$. How does this graph help explain the domain of the graph in Q6 c) ?

Activity 9 Gradient of composite functions

Aim: Verify the chain rule. Link the chain rule to composite functions.

1. Consider the function $y = \left(x^2 - 3x + 1\right)^3$.
   This can be thought of as a composite function, with the inner function a quadratic and the outer function cubic.

   a) Write out the expansion of $\left(x^2 - 3x + 1\right)^3$.
   (Use ClassPad - [Action | Transformation | expand])

   b) Differentiate this expression.
   (diff(ans))

   c) Factorise the result.
   (factor(ans))

   d) Your answer should have three factors.
   What connections can you make between the factors and the original function?

   e) Summarise the result as a conjecture about the derivative of composite functions.

2. Test your conjecture with other examples. Fill in the table by

- Making up an example of a composite function
- Using your conjecture from Q1 e) to predict the derivative
- Check this using ClassPad

Composite function	Derivative predicted by your conjecture.	Verified on ClassPad, Yes or No.

3. For each derivative function determine an original function and check your answer by differentiating (effectively using Guess and Check).

a) $4\left(x^2+6x-1\right)^3(2x+6)$

b) $(6x+5)e^{3x^2+5x}$

c) $\cos\left(x^3-7\right)\times\left(3x^2\right)$

d) $-2(3x+4)^{-3}\cdot 3$

e) $\frac{1}{2}\left(x^2-4\right)^{-\frac{1}{2}}\cdot 2x$

f) $(8x+24)\sin\left(x^2+6x-1\right)$

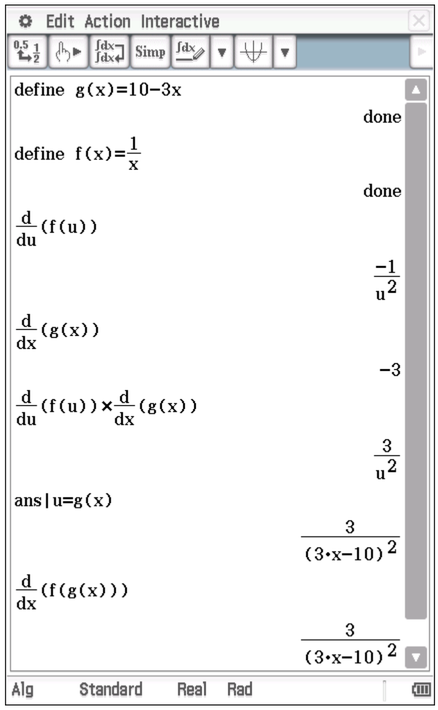
4. Complete the table.

Composite function $y = f(g(x))$	Inner function $u = g(x)$	Outer function $y = f(u)$	$\dfrac{dy}{du}$	$\dfrac{du}{dx}$	$\dfrac{dy}{du} \times \dfrac{du}{dx}$	$\dfrac{dy}{dx}$
$\dfrac{1}{10-3x}$	$10-3x$	u^{-1}	$-\dfrac{1}{u^2}$	-3	$\dfrac{3}{u^2}$	$\dfrac{3}{(10-3x)^2}$
$\sqrt{10-3x}$		$u^{\frac{1}{2}}$				
$(x^2-9)^3$						
e^{x^2-9}						
$\sin(x^2-9)$						
$e^{3\sin x}$						

Verify the table entries using ClassPad
Duplicate the calculations for $y = f(g(x))$ in Main.

- Define the two functions
- Calculate $\dfrac{dy}{du}$
- Calculate $\dfrac{du}{dx}$
- Calculate $\dfrac{dy}{du} \times \dfrac{du}{dx}$
- Substitute for u: $g(x)$
- Compare to $\dfrac{d}{dx} f(g(x))$

For the remaining rows in the table go back and edit the function definitions.



EXTENSION

5. Use the chain rule (twice) to differentiate $\sqrt{10-\dfrac{3}{x^2-9}}$

Learning notes

It is likely that the results displayed by ClassPad and what you do by hand will look different. Some algebraic manipulation will be required to check that the different expressions are the same expression. Using CAS commands like simplify, expand, factor and combine can be helpful when you are checking answers.

Q1 will be revision if you are familiar with the chain rule. Try to develop an intuitive approach to calculating the derivative of composite functions. In Q3 look for the result of using the chain rule.

Q4 You can check your working by simply redefining the functions at the beginning of the Main screen.

Q5 requires use of the chain rule two times. Perhaps a reason for the name "chain" rule?

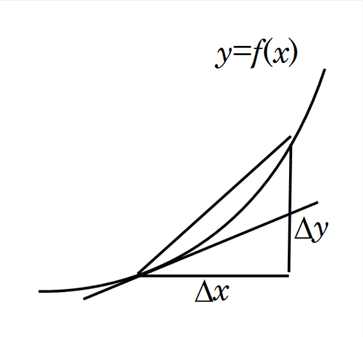
The following argument may help to **justify the chain rule.**

The slope of the chord $\dfrac{\Delta y}{\Delta x}$ gets closer and closer to the slope of the tangent as the chord gets shorter i.e $\dfrac{dy}{dx}=\lim\limits_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x}$.

For $y=f\left(g(x)\right)$ if we let $u=g(x)$ it follows that

$$\frac{\Delta y}{\Delta x}=\frac{\Delta y}{\Delta u}\times\frac{\Delta u}{\Delta x}$$

and
$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta u}\times\frac{\Delta u}{\Delta x}\right)$$
$$=\lim_{\Delta u\to 0}\frac{\Delta y}{\Delta u}\times\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}\quad\text{(a limit theorem)}$$
$$=\frac{dy}{du}\times\frac{du}{dx}$$



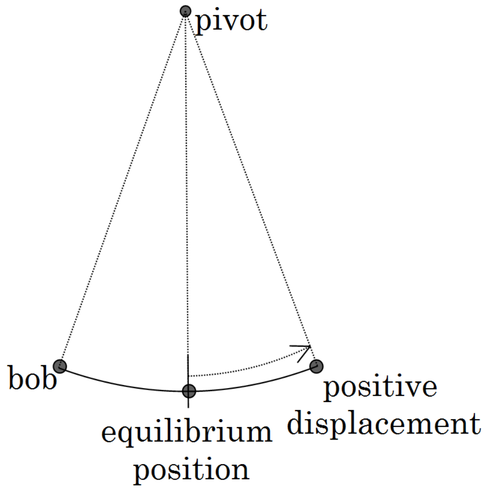
Activity 10 Pendulum motion

Aim: Use trigonometric functions to model the motion of a simple pendulum.

The bob of a pendulum is pulled back and released. It is free to swing back and forth and its displacement d cm relative to its equilibrium position at time t seconds after release is measured and given in the table below.

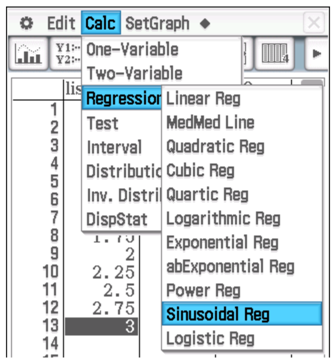
t	0	0.25	0.5	0.75	1	1.25
d	10.0	4.5	–6.0	–9.8	–2.8	7.3

1.5	1.75	2	2.25	2.5	2.75	3
9.4	1.1	–8.4	–8.6	0.74	9.3	7.5

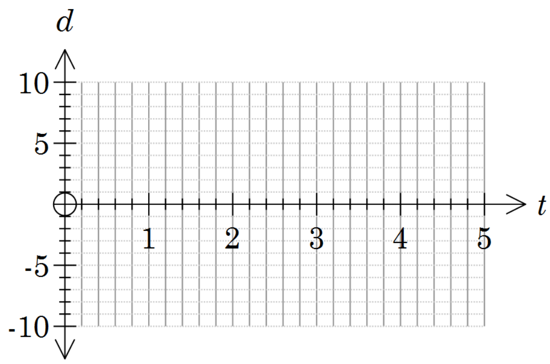


Create a model

- Select [⚙ | Basic Format] and change
  Number Format to Fix 3
  Angle to Radian
- Open the Statistics app
- Enter time values in list1 and displacement in list2
- Select [Calc | Regression | Sinusoidal Reg]
- Copy formula to y1 for recall in Main



1. a) Write down the sinusoidal model for the displacement d cm in terms of the time t seconds and sketch the graph.



   b) Determine the period of the motion. (Recall that period $T = \frac{2\pi}{b}$)

   c) Predict the displacement of the bob at $t = 4$ s.

d) Hence determine the total distance travelled by the bob in the first 4 seconds.

2. a) Differentiate the displacement equation to determine an equation for the velocity of the bob at time t seconds.

b) What is the maximum velocity and when does this first occur?

c) What is the displacement of the bob when maximum velocity occurs?

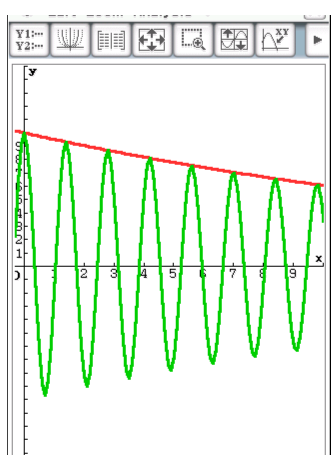
d) What is the displacement of the bob when it is stationary?

3. Determine the displacement of the bob when its acceleration is zero.

EXTENSION

The model developed above in this activity assumes no friction, hence perpetual motion of the pendulum. In reality, friction slowly reduces the amplitude until the pendulum is permanently stationary

We can model the "damping" effect of friction by multiplying the trigonometric equation by an exponential decay function. Look at the maximum displacement decay slightly with each swing shown in the screenshot.



The pendulum maintains a period of ~1.4 seconds, and is released from a displacement of 10 cm. After 4 complete swings (at t = 5.6 s), the maximum displacement has reduced to 7.6 cm.

4. The damped motion can be modelled by an equation of the form $d = Ae^{kt}\cos(bt)$

   a) Explain why:

      (i) a cosine function has been chosen for the periodic part of the equation

      (ii) A = 10 cm

      (iii) $b \approx 4.5$

   b) Use the displacement at $t = 0$ and $t = 5.6$ to determine a value for k and hence write down the model for the damped displacement.

   c) Hence predict the displacement and velocity of the bob as it completes its 10th swing.

Learning notes

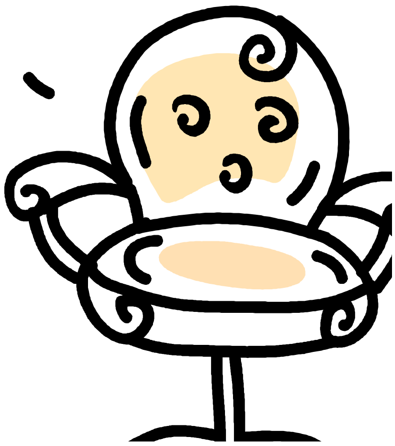
Q1 b) b is the coefficient of t in the equation.

Q4 b) An exponential decay model for the maximum displacement can be calculated in Statistics using co-ordinates (0, 10) and (5.6, 7.6). Alternatively, given we have A = 10, we can solve for k in Main since $d = 7.6$ when $t = 5.6$.

Activity 11 Comfy chairs

Aim: Solve optimisation problems using calculus.

1. Comfy chairs have a factory capable of producing up to 250 chairs per week.



   - Suppose x chairs are made and sold each week and the selling price per chair is set at $240+\dfrac{600}{x}$ dollars.
   - The cost of materials and labour is $42x+12x^{\frac{3}{2}}$ dollars per week.
   - There are also fixed costs of $2400 each week (rent etc.).

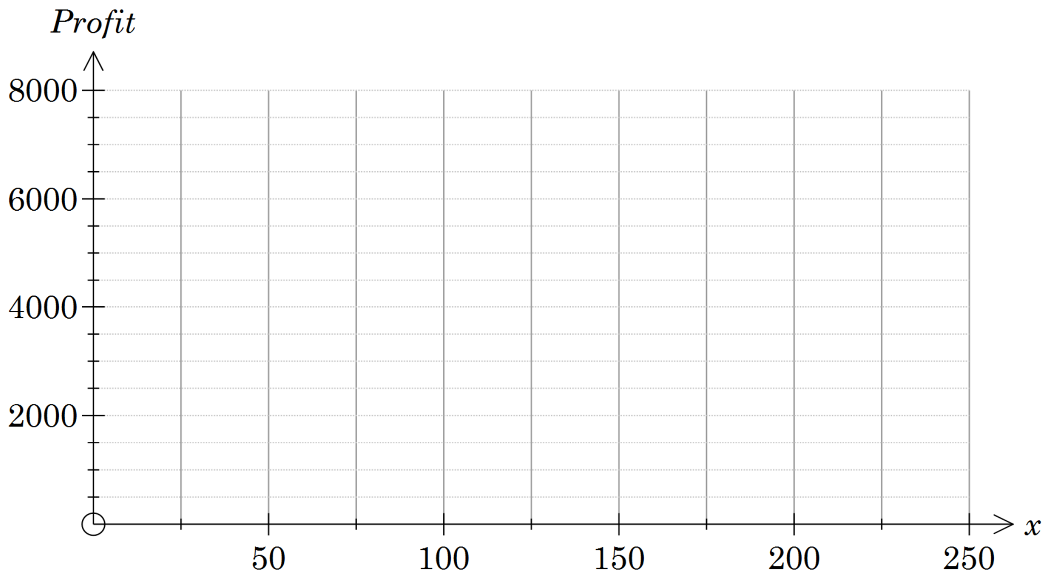
a) What is the domain?

b) Show that the manufacturer's weekly profit is $P=-12x^{\frac{3}{2}}+198x-1800$

c) Use calculus to determine how many chairs should be made and sold each week to maximise the profit.

d) What is the maximum weekly profit?

e) Sketch a graph of the profit function, showing the features from parts a) to d).



2. The Deluxe Comfy chair costs more to make and can be sold at a higher price. For this chair:

- 200 is the maximum number of chairs that can be made in a week;
- the selling price is $325+\dfrac{600}{x}$ dollars per chair; and
- the materials and labour costs are $55x+12x^{\frac{3}{2}}$ dollars per week, and fixed costs of $2400 remain.

How many chairs should now be made to maximise the profit if Comfy chairs switch to producing their Deluxe model? Write a full solution.

Learning notes

Where an approximate solution is required, a graphical approach is appropriate for optimisation problems. However, this problem asks for the use of calculus. Using CAS enables you to demonstrate your knowledge of the required steps and ability to interpret the results in the context of the problem. CAS is particularly useful when the algebra is complex or beyond our current skills.
To show that you have used calculus in such optimisation problems, your working should show:

- the function;
- the derivative;
- the equation to be solved to find the stationary points;
- the nature of the stationary points and a justification;
- a consideration of the end points, as the end points could be the optimum; and
- the answer, stated in the context of the problem.

Q1 is more typical of exam questions in that the method is set out. It is also possible to do later parts of the problem even if you are unable to establish the relationships.

For Q2 the CAS solution can be most simply done by editing your working on ClassPad. Ensure you write out all the steps outlined above in your solution.

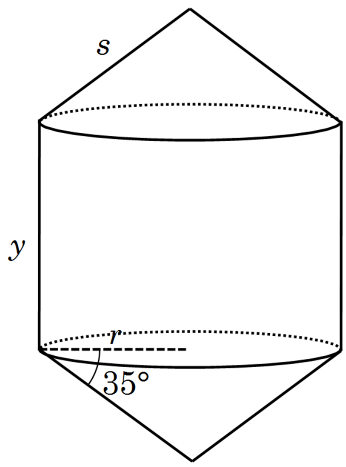
Activity 12 Silos'r'us

Aim: Use calculus to optimise dimensions of a silo for minimum cost.

Silos'r'us

Sam, the manager of Silos'r'us, notes that price per volume is an important part of the customer's decision making. Can he get the same volume but make it cheaper?

The picture shows a fertiliser silo which is cylindrical with a conical top and bottom. The cones have an angle between base and side of 35°.



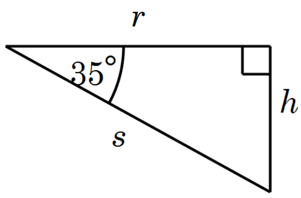
Source: http://www.ahrens.com.au/products/agri/silos/transportable-silos/fertiliser-silos

Volume formulae	**Surface area** formulae (curved parts)
Cylinder $V = \pi r^2 h$	Cylinder $A = 2\pi rh$
Cone $V = \frac{1}{3}\pi r^2 h$	Cone $A = \pi rs$

Note: Values in formulae are rounded to three significant figures.

1. Show that:

   a) the height of a conical end is $h = r\tan 35°$



   b) the slant height of a conical end is $s = \dfrac{r}{\cos 35°}$

2. Show that the volume of a tank with base radius (r) of 1.7 m and height (y) of 3.1 m is 35.4 m^3 (3 s.f.).

3. Another model has a volume of 15 kL.

a) Calculate the height (y) if the base radius is 1.25 m.

b) Show that the height $y \approx \frac{4.77}{r^2} - 0.467r$ m.

c) State the domain of the height function in b), i.e. the possible radii of the silo.

d) Show that surface area of the sides (cylindrical part) is $-2.93r^2 + \frac{30}{r}$ m^2.

e) Show that surface area of the top (and bottom) is $3.84r^2$ m^2.

The cost per unit area of the top cover is 50% more than the cost per unit area of the sides and the cost of the base is double the cost per unit area of the sides.

f) Show that the cost of making the tank is $C = k\left(\dfrac{30}{r} + 10.49r^2\right)$ where k is the cost per unit area of building the cylindrical sides of the silo.

g) Use calculus to determine the value of r that minimises the cost.

EXTENSION

4. Sam wants a formula to enable him to calculate the dimensions for any tank that minimise the cost per unit volume, i.e. a formula for the radius and height of a tank in terms of V, t and b where

   - V is the volume
   - t is the ratio of the cost/ unit area of making the **top** compared to the sides and
   - b is the ratio of the cost / unit area of making the **bottom** compared to the sides.

   Give Sam the formula and then present a full solution in support.

Learning notes

Most of the expressions are decimal approximations so it is easier if you put ClassPad into Decimal mode.

Setup

- Set to Decimal mode.

Set display to 3 significant figures.

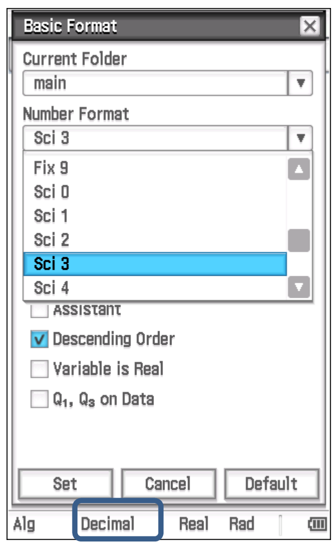
- Select [⚙ | Basic Format]
- Choose Sci3

Or

Set display to 3 decimal places.

- Select [⚙ | Basic Format]
- Choose Fix3

The setting can be changed as you work through the problem.



This activity provides an opportunity to develop proficiency with CAS. While most of the working can be done using pen and paper methods, using CAS can enable a greater focus on what you are trying to do with less emphasis on the manipulations. It is still necessary to interpret the output.

The more complex test questions will often have a similar structure to this investigation. In order to solve the problem, small steps to establish the relationship (or formula) are asked first. It is not necessary to justify these statements in order to move to the next part of the problem and show understanding and an ability to use the result in later parts.

Q3 c) asks for the domain. The radius and height must both be greater than 0. Solve the expression for height = 0 to determine the maximum radius.

Q4 asks you to repeat the steps more generally. In a real situation this would be the problem.

Chapter 2 Integrals

Activity	ClassPad applications	Key concepts
What might the function be?	Main	Appreciate integration as the inverse of differentiation Determine rules of integration through guess and check methods
Are we there yet?	Spreadsheet Program	Area under the velocity-time graph representing distance travelled
The fundamental theorem of calculus	Program Main	Approximate areas by summing rectangles, link to definite integral
Integrate	Main	Become familiar with the syntax and options of the integrate command
Distance from acceleration	Main	Determine velocity and distance functions given acceleration
Tax time	Main Graph&Table	Apply integration concepts in a novel context

$$\int_{Calculus}^{Let's} du$$

Activity 13 What might the function be?

Aim: Appreciate integration as the inverse of differentiation.
Determine rules of integration through guess and check methods.

1. Complete the list below by differentiating the given function or guessing a function that will produce the given derivative function

	Function	Derivative
a)		3
b)		$2x+3$
c)	x^3-3x^2	
d)	$x^3-3x^2+14.7$	
e)		$3x^2+6x$
f)	$\frac{1}{5}x^5$	
g)		x^4+3
h)	e^{2x}	
i)		$2e^{2x}+4$
j)	$\sin x$	
k)		$-\sin x$
l)	$\cos\left(7x^3\right)$	
m)		$21x^2\sin\left(7x^3\right)$
n)	xe^x	
o)		$(x+1)e^x$
p)		$\cos 5x$
q)		e^{4x+3}

Check your answers by differentiating the left hand column using ClassPad.

2. Determine a function that has the derivative shown and passes through the given point. Check your answers by differentiating and substituting.

	Function	Derivative	Function passes through point
a)		$x^3 + 2$	(0, 6)
b)		$e^x - 7$	(0, –3)
c)		$4\cos 4x$	(0, –3)
d)		e^{x+2}	(–2, 2)
e)		$\frac{1}{2}x^{-\frac{1}{2}}$	(9, 2)
f)		$6\sin\left(3x + \frac{2\pi}{3}\right)$	$\left(\frac{\pi}{9}, 7.3\right)$

3. Determine all possible functions that have the derivative shown and check your answers by differentiating.

	Function	Derivative
a)		x^n
b)		e^x
c)		$\sin x$
d)		$\cos x$
e)		$\sin x + 2\cos x$
f)		$(ax+b)^n$
g)		e^{ax+b}
h)		$\cos(ax+b)$

Learning notes

Since guess and check is the approach, you should check your guesses by differentiating in Main.

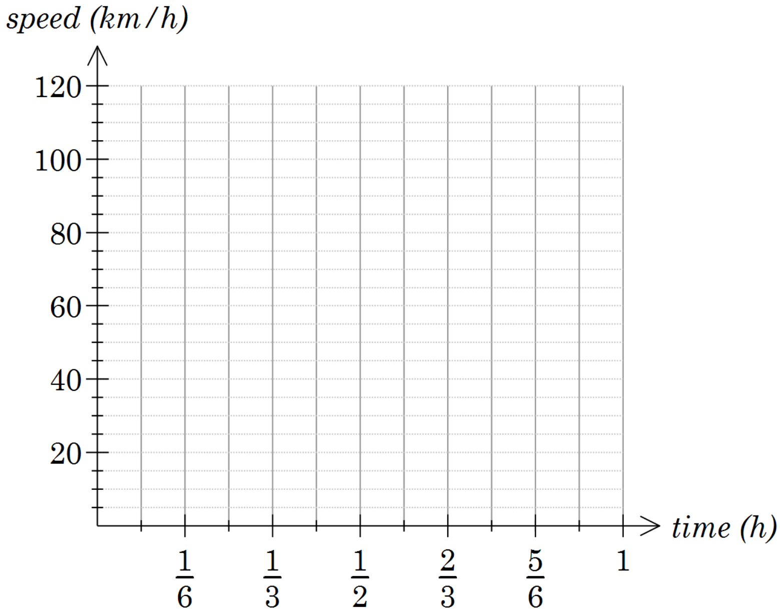
Activity 14 Are we there yet?

Aim: Develop the concept of area under the velocity-time graph representing distance travelled.

1. Menelaus was driving out of the city. He noted his speed (from the speedometer) at several times. The results are shown in the table below.

   a) Plot the results on the graph.

Time	Speed (km/h)
1:00	0
1:05	57
1:10	50
1:15	61
1:20	55
1:25	80
1:30	75
1:40	98
1:50	101
2:00	99

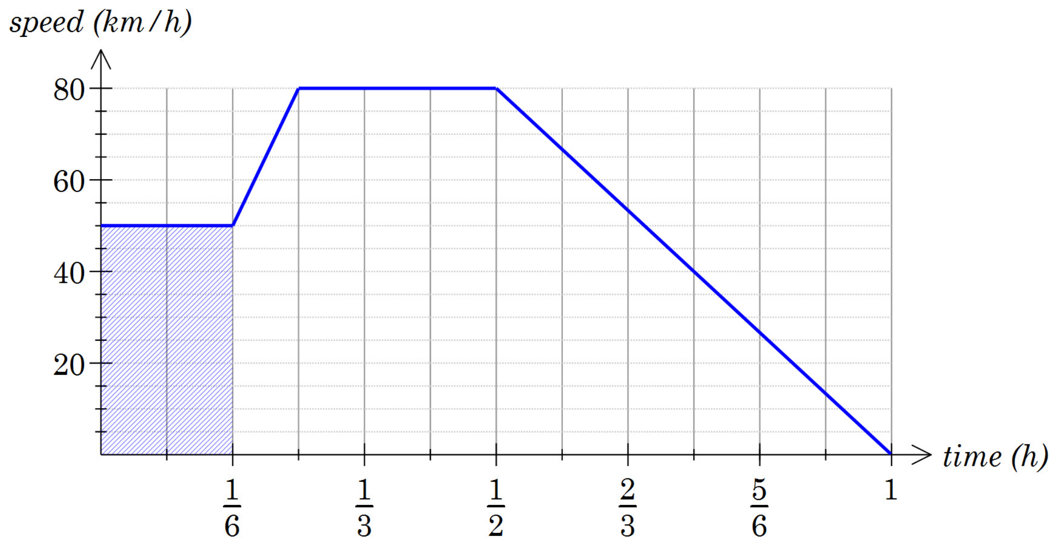


   b) Estimate Menelaus' average speed

   c) Estimate the distance Menelaus travels in the hour (average speed × time)

   d) What assumptions (approximations) are you making to arrive at your estimate of the distance travelled?

2. This graph might represent another trip.



a) Consider the interval 0 to 10 minutes ($\frac{1}{6}$ hour):

(i) What is the distance travelled?

(ii) What is the area under the curve (i.e. the shaded rectangle)?

b) Why are parts i) and ii) the same calculation?

c) Consider the interval 10 to 15 minutes, calculate

(i) the average speed

(ii) the area of the trapezium, i.e the distance travelled

d) Consider one “square” on the grid above. What is the:

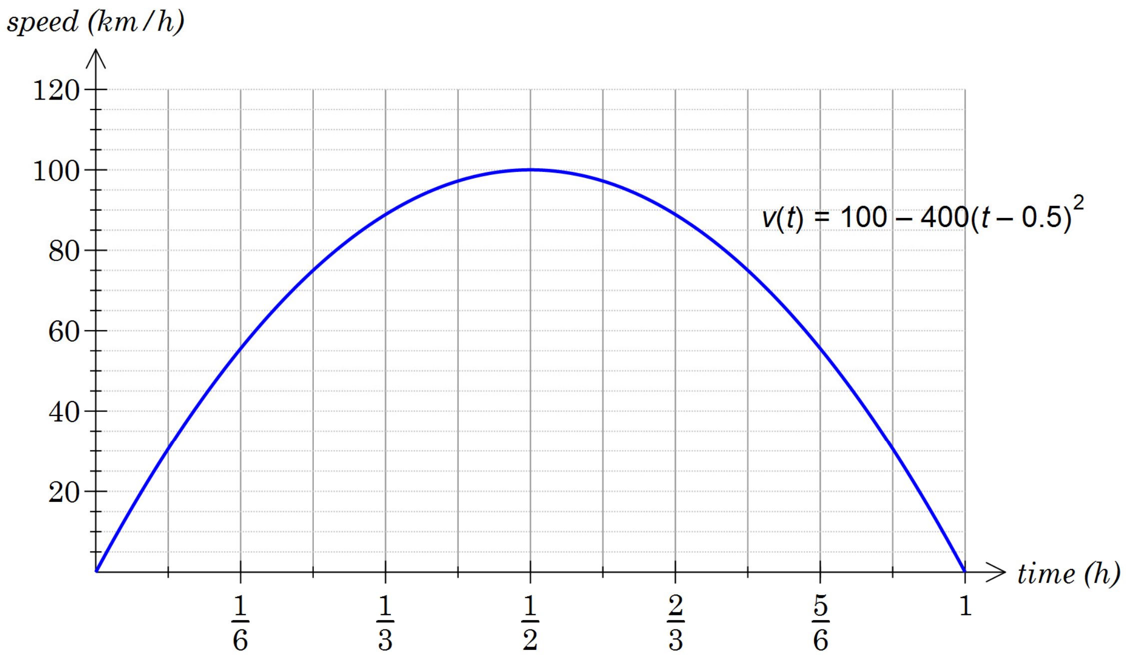
(i) width (in hours)?

(ii) height (in km/h)?

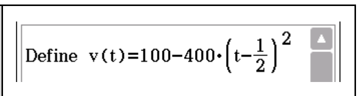
(iii) area and what are the units?

e) Calculate the total distance travelled in the hour

3. What if the velocity–time graph is curved e.g. like in the graph below? An estimate can be generated by breaking the graph up into segments. Calculate the area under the graph to determine the total distance travelled.



a) The velocity is modelled by the equation $v(t) = 100 - 400(t - 0.5)^2$.

Define the velocity function in Main	

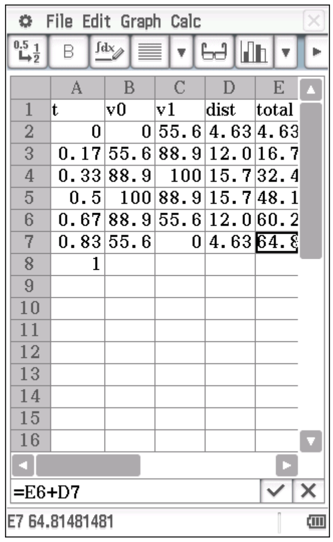
Complete the table by calculating the speed at the beginning of the time interval, the speed at the end of the time interval, the average speed over the time interval and the distance travelled.

Time (min)	v_{start} (km/h)	v_{end} (km/h)	average v (km/h)	distance travelled (km)
0 – 10	$v(0) = 0$	$v\left(\frac{1}{6}\right) = 55.6$	27.8	4.6
10 – 20	56			
20 – 30				
30 – 40				
40 – 50				
50 – 60				

b) Duplicate the table in the Spreadsheet app to estimate the total distance travelled.

Create Spreadsheet

- Open Spreadsheet app
- Put in column headings
- Put 0 in cell A2
- Formulae:
  A3: =A2+1/6
  B2: =v(A2)
  C2: =v(A3)
  D2: =(C2+B2)/2×($A$3–$A$2)
- Copy formulae down: columns A to D
- Add total distance
  E2: =D2
  E3: =E2+D3 and copy down
- Save the spreadsheet



	A	B	C	D	E
1	t	v0	v1	dist	total
2	0	0	55.6	4.63	4.63
3	0.17	55.6	88.9	12.0	16.7
4	0.33	88.9	100	15.7	32.4
5	0.5	100	88.9	15.7	48.1
6	0.67	88.9	55.6	12.0	60.2
7	0.83	55.6	0	4.63	64.8
8	1				

c) With more regions (i.e. thinner intervals) we should get a better estimate. Modify your spreadsheet and estimate the distance travelled if there are 10 regions.

d) Describe what is happening to the area (our estimate for the distance travelled) when the number of intervals increases.

e) Is the actual area more or less than our estimates? Give reasoning to justify your answer?

EXTENSION

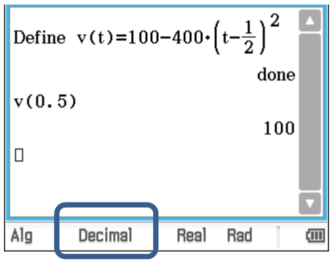
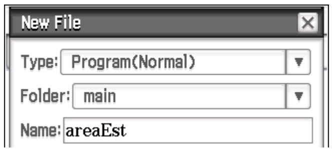
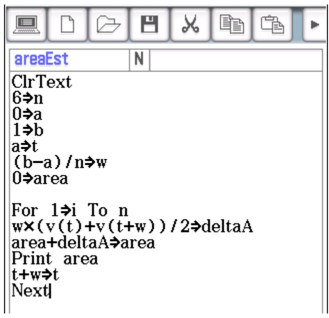
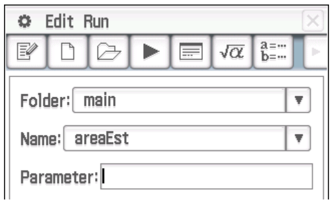
f) Modify your function in Main and then estimate the distance travelled in the first hour if the velocity is modelled by:

(i) $v(t) = 100(1 - e^{-2t})$

(ii) $100\sin\left(\dfrac{\pi t}{2}\right)$

Modify the velocity function in Main
Return to the Spreadsheet and recalculate
[File | Recalculate]

4. To investigate the effect of reducing the width (increasing the number) of the regions, the ClassPad spreadsheet is not the easiest tool. We are now going to write a program to make it easier to investigate. If you have access to the website you can download the program and import it into your ClassPad.

Setup • In Main ensure ClassPad is in Decimal mode • Define the velocity function [Interactive | Define]	
Open Program app This is likely to be on the second page of applications (swipe left on the menu screen)	
Open a new file • Tap or [Edit | File New] • Give a name e.g.areaEst • Tap OK	
Enter the program • Tap to open the program editor • Enter the code as shown You can use the keyboard for everything but it is less error prone to use the menus for commands e.g. [I/O | Clear | ClrText] [Ctrl | For | For] • Tap to save	
Run the program • Tap to save • Tap • Tap to run the program	
Edit the program • Tap to open the program editor	

It doesn't work!

- It is likely you will make a mistake in entering the code. If you can't save ClassPad will place the cursor on the line which is problematic. Check the code, find and correct the error and try to save again.
- If it runs but the output isn't as expected:
  Tap to return to the edit page, check the code, find and correct the difference then rerun the program

a) Check that you get the same answers as your spreadsheet.

b) Explore what happens as the number of intervals increases
Alter the value of n in the program and run it again. Record your results in the table:

Number of intervals	Total distance

c) Determine the distance travelled accurate to 1 decimal place

d) Use your program to estimate the distance travelled in the intervals

(i) $0 \le t \le 0.25$

(ii) $0.25 \le t \le 0.5$

(iii) $0 \le t \le 0.5$

(iv) $0.25 \le t \le 0.75$

5. a) Verify that the derivative of $s(t) = 100t - \frac{400}{3}\left(t - \frac{1}{2}\right)^3$ is $v(t)$

a) Complete the table of values for $s(t)$

t	0	0.25	0.5	0.75	1
$s(t)$					

b) How might these values be connected to your answers to Q4 c)?

6.

a) Use your program to compete the table.

	Estimate of distance travelled		
Velocity function	In first hour	In second hour	In first two hours
$v(t)=100(1-e^{-2t})$			
$v(t)=100\sin\left(\frac{\pi t}{2}\right)$			

b) Determine anti-derivatives for the velocity functions

(i) $v(t)=100(1-e^{-2t})$

(ii) $v(t)=100\sin\left(\frac{\pi t}{2}\right)$

c) Calculate the values of the anti-derivatives for t = 0, 1 , 2

Velocity function	Anti-derivative	$t=0$	$t=1$	$t=2$
$v(t)=100(1-e^{-2t})$				
$v(t)=100\sin\left(\frac{\pi t}{2}\right)$				

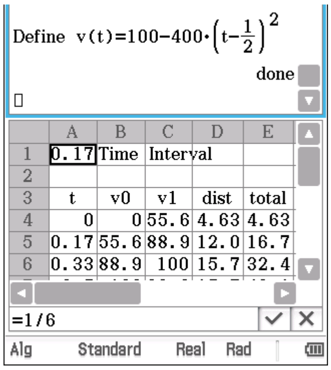
d) Explain how these values can be used to determine the distance travelled in the second hour column of part a).

Learning notes

It is assumed that you are sufficiently comfortable with creating spreadsheets to not require detailed instructions.

The screenshot shows a modified spreadsheet that makes it easier to change the time interval, only cell A1 needs to be changed

- Insert 2 extra rows
- Enter the width of each region and a label
- Delete column C
- Formulae:
  C4: =(B4+B5)/2×$A$1
- Copy formulae down in columns A to D to row 28
- Read the total distance in column D on the row before the time is 1.



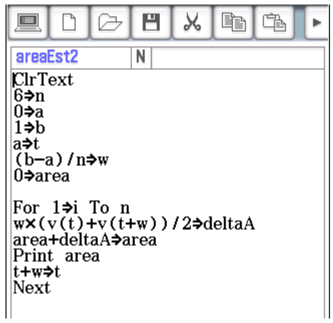
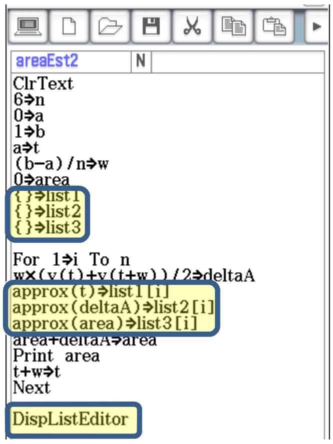
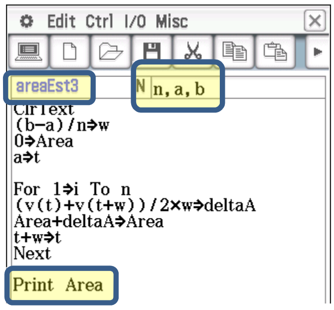
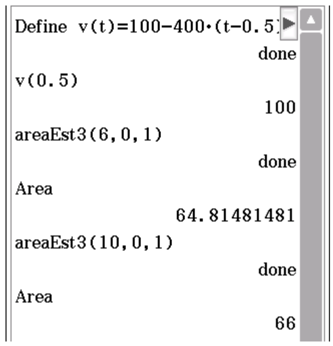
A computer-based spreadsheet works well. It is much easier to extend the spreadsheet with more rows and adding if statements into the formulas can improve the presentation. However, you will have to put the full formula into the cells to calculate the velocity as you are unlikely to be able to define the velocity function.

About the program:

The program has two main sections:

Code	**Explanation**
Initialisation	
ClrText	Clears the text window ready for output
	Store variables:
6⇒n	n – number of regions
0⇒a	a – start time
1⇒b	b – end time
a⇒t	t – current time value
(b–a)/n⇒w	width of the region (time)
0⇒area	location for storing the distance
Main loop	
For 1⇒i To n	Start loop i will take values from 1 through to n
(v(t)+v(x+t))/2×w⇒deltaA	deltaA is the distance (av speed x time)
area + deltaA⇒area	add to area
Print Area	display cumulative area
t+w⇒t	increment to next time interval
Next	end of loop

Once the program is working then modifications can be made. If you have the opportunity try out the following:

Create a new modified program • Tap to open the program editor • [Edit | Select All] • [Edit | Copy] • Tap to exit the program editor • Tap to create a new file with a new name e.g. areaEst2 • Open the program editor • [Edit | Paste] • Adjust the program code as desired	
Display calculations as a list • Remove the ClrTxt and Print commands, for neatness New commands are highlighted • Clear lists 1 to 3 • Store values for time in list1, area of region in list2 and running total in list3 • Use approx() to ensure output is a decimal • DispListEditor – displays the Statistics lists window where the output is stored.	
Introduce parameters This version reduces the number of lines in the program. Parameters are used and we can run the program in Main. The only output is the total distance displayed at the end	
In this screenshot the program is being run in Main. The use of parameters means that we can run the program again with different inputs without needing to go in and edit the program.	

Activity 15 The fundamental theorem of calculus

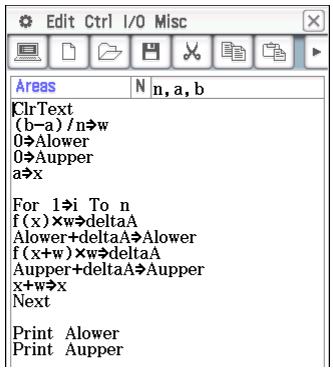
Aim: Approximate areas by summing rectangles, link to definite integral.

This activity builds on the programming done in the previous activity.

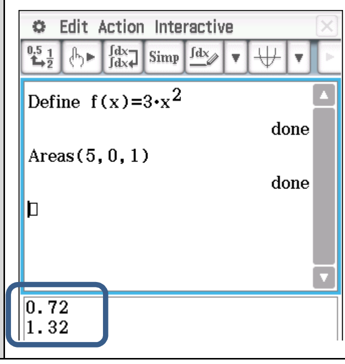
1. Enter and save the program shown

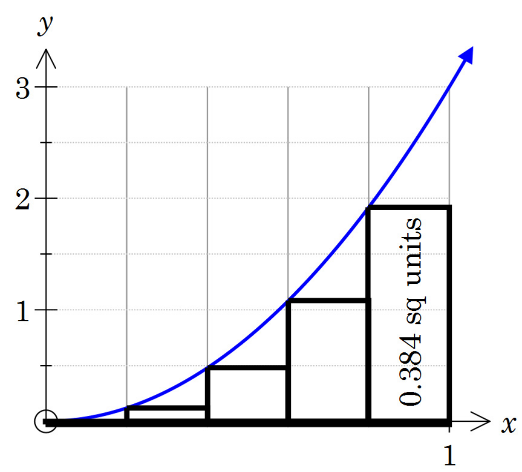
You can copy and edit a program from the previous activity

- Open the program
- Select all the text
- [Edit | Copy]
- Tap [new file icon] to open a new file
- Name it *Areas*
- [Edit | Paste]
- And then edit as appropriate

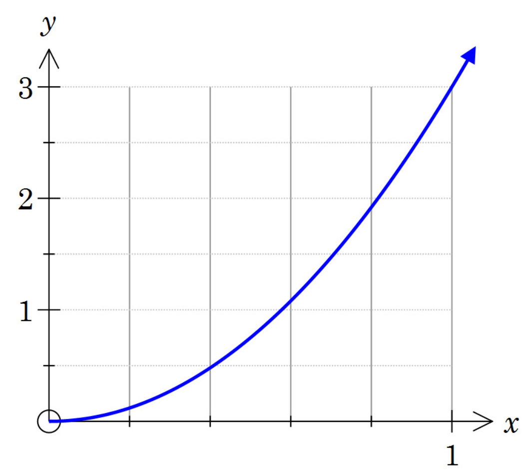


This screenshot shows the program being run in Main. The use of parameters means that we can run the program again with different inputs without needing to go in and edit the program. When Areas(5,0,1) is executed the output is displayed in the program output window. The precise area lies between the two values, providing the function is increasing (or deceasing) in the region.





Graph 1



Graph 2

a) Charlotte is attempting to justify the value of 0.72 and draws the rectangles on the graph. Explain how **Graph 1** helps explain the value of 0.72.

b) Draw rectangles on **Graph 2** to show what the value of 1.32 represents.

c) Use your program to complete the table

Area under the curve $y = 3x^2, 0 \le x \le 1$ i.e. between the curve, $x = 0$, $x = 1$ and the x-axis		
Number of intervals	Lower bound	Upper bound
5	0.72	1.32
10		
50		
100		
1000 (This can take a long time on your ClassPad)		

d) As the number of intervals increases what do you notice about the two areas?

e) Make a prediction for the precise area under the curve.

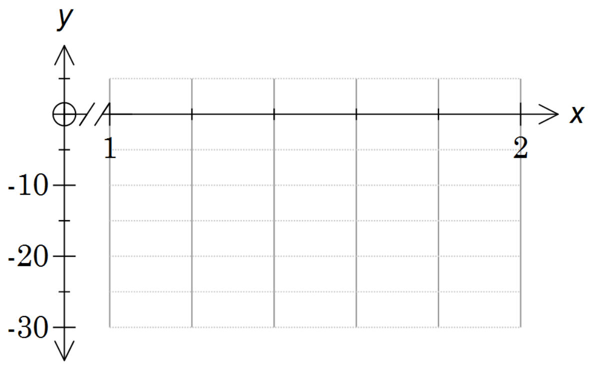
2. a) Use your program to predict values for the intervals shown

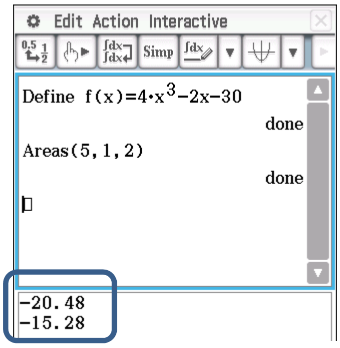
Area under the curve $y = 3x^2$	
Interval	Area
$0 \le x \le 2$	
$0 \le x \le 5$	
$2 \le x \le 5$	
$3 \le x \le 10$	
$-1 \le x \le 1$	

b) How is the area related to the antiderivative (x^3) of $3x^2$?

3. Consider the function $f(x) = 4x^3 - 2x - 30, \ 1 \le x \le 2$

a) Draw the graph on the grid below.





b) Use the graph to explain the calculator output shown.

c) State the area under the graph for the interval $1 \le x \le 2$

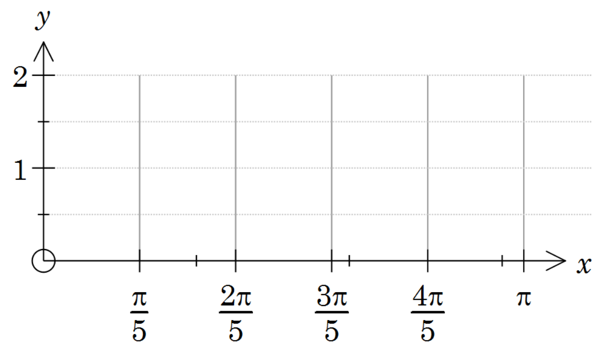
d) How does the area relate to the antiderivative of $f(x)$?

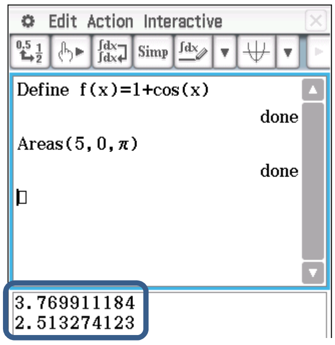
e) Use your program to predict values for the intervals and function given:

Area under the curve $y = 4x^3 - 2x - 30$	
Interval	Area
$0 \le x \le 1$	
$0 \le x \le 2$	
$0.5 \le x \le 1.5$	

4. Consider the function $f(x)=1+\cos x,\ \ 0\le x\le\pi$

a) Draw the graph on the grid below.





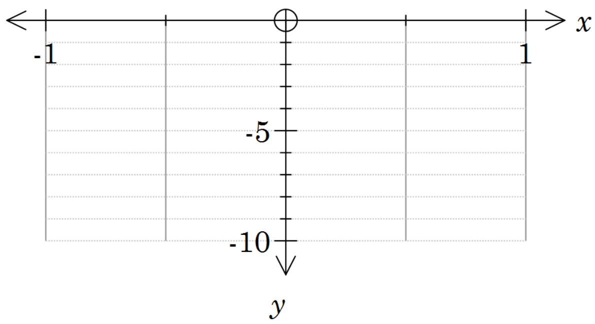
b) Use the graph to explain the calculator output shown.

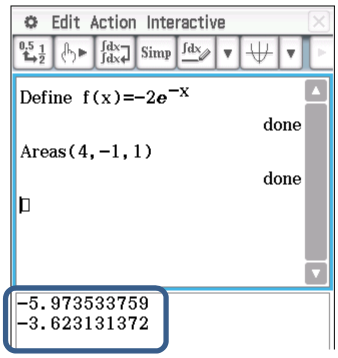
c) State the area under the graph for the interval $0\le x\le\pi$

d) How does the area relate to the antiderivative of $f(x)$?

5. Consider the function $f(x)=-2e^{-x},\ \ -1\le x\le 1$

a) Draw the graph on the grid below.





b) Use the graph to explain the calculator output shown.

c) State the area under the graph for the interval $-1\le x\le 1$

d) How does the area relate to the antiderivative of $f(x)$?

6. Summarise your results by describing how the area is estimated using "lower" and "upper " rectangles

Function is		
positive or negative	increasing or decreasing	Area estimate is calculated by
$f(x) \geq 0$	$f'(x) \geq 0$	
$f(x) \leq 0$	$f'(x) \geq 0$	
$f(x) \geq 0$	$f'(x) \leq 0$	
$f(x) \leq 0$	$f'(x) \leq 0$	

Learning notes

Refer to the previous activity to see details on how to write, store and run a program.

Q3 Edit the function in Main and then rerun the program

To be sure that the area lies between the two values calculated by the program then the function must be the same sign throughout the interval and either increasing or decreasing in the interval. If not then split the interval into pieces where this is the case.

When using CAS, using the absolute value function will ensure the function does not change sign, i.e. $|f(x)|$. This technique doesn't simplify working when using by-hand methods.

Activity 16 Integrate

Aim: Appreciate different ways of accessing ClassPad's integrate command. Become familiar with the syntax and options of the integrate command.

1. Set up ClassPad

Setup

- Open Main
- Check ClassPad is in Standard mode

Clear variables

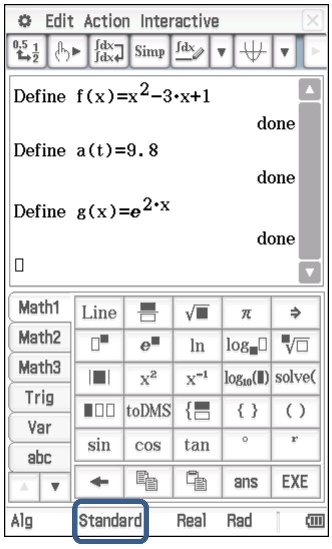
- Select [Edit | Clear All Variables] and tap OK. (Functions and Lists are not cleared)
- Use Variable Manager to clear functions and lists. [⚙ | Variable Manager]

Define functions as shown

- [Interactive | Define]

Enter expressions

- Select [Action | Calculation | ∫] for the integrate command
- Select [Action | Calculation | diff] for the differentiate command



a) Enter each of the commands listed in the table and record the ClassPad output. Where possible explain what the function is doing.

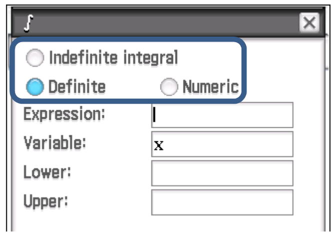
Command	Output	Explanation
∫(f(x))		
∫(diff(f(x),x))		
∫(diff(f(x),x,1,3))		
∫(f(x),x)		
∫(f(x),x,0,1)		
∫(f(x),x,0,2)		
∫(f(x),x,0,r)		
∫(f(x),x,l,r)		
diff(∫(f(x),x,0,r),r)		

b) Summarise the $\int$ command and its syntax.

2. Use ClassPad and the interactive menu to calculate each expression in the table. Record the result in appropriate mathematical notation. Don't forget the constant!

Enter expressions

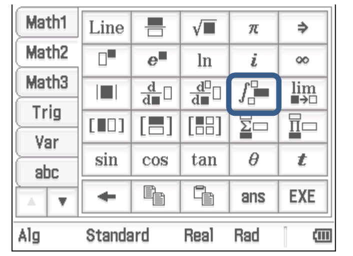
- Select [Interactive | Calculation | $\int$] for the integrate command
- Select Definite when required



Command	Output
$\int x^3 dx$	
$\int g(x)dx$	
$\int x^3 + g(x)dx$	
$\int a(t)\ dt$	
$\int \left(\int a(t)\ dt \right) dt$	
$\int_0^{10} a(t)\ dt$	
$\int_0^{10} x^3\ dx$	
$\int \left(\frac{d}{dx} g(x) \right) dx$	
$\frac{d}{dx} \left(\int g(x)dx \right)$	
$\int g(x^2 - 7.2x) \times 2 \times (2x - 7.2) dx$	
$\int 25y^4 - 12y^2 - 1\ dy$	
$\int g(f(x)) f'(x) dx$	

3. Use template to answer the following.

- Press Keyboard
- Tap [∫ template] from the Math2 tab



a) $\int_1^2 4.68x\ dx$

b) $\int_2^{12} 4.68x\ dx$

c) $\int_1^{12} 4.68x\ dx$

d) $\int_{-2}^2 3x^3 - 8x\ dx$

e) $\int_0^5 3\sqrt{x}\ dx$

f) $\int_5^0 3\sqrt{x}\ dx$

g) $\int_0^5 -\left(3\sqrt{x}\right)\ dx$

h) $\int_0^1 xe^{x^2} dx$

4. State any properties of definite integrals that you have verified by example in Q3.

Learning notes

In this activity you use different ways of calculating integrals. Most of the time the template is easiest as it mirrors the notation we use when writing integral expressions. To calculate an indefinite integral with the template just leave the limits blank.

Activity 17 Distance from acceleration

Aim: Determine velocity and distance functions given acceleration.

1. Constant acceleration

   Assume the acceleration of a free falling object on the surface of a planet is the constant g. An object is thrown upward with a launch velocity of 25 m/s from a cliff 23 m high.

   a) Fill in the table showing the information given. Pay attention to the direction and signs.

Acceleration	
Height at time $t=0$	
Velocity at $t=0$	

   b) Integrate to determine the velocity function.

   c) Integrate again to determine a function for the height at time t.

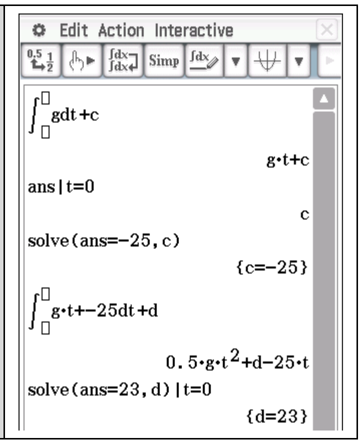
   d) How long will the object take to reach the bottom of the cliff assuming $g = 9.8ms^{-2}$?

Duplicate by-hand working

- Clear all variables
- Press Keyboard

Calculate integral

- Tap from the Math2 tab.



e) Determine the displacement function for an object moving in a straight line with initial velocity u and initial position s_0 under constant acceleration a.

2. Variable acceleration
The acceleration of a weight on the end of a spring is given by $a(t) = 40\sin 2t$ cms^{-2} with initial velocity –20 cms^{-1} and initial displacement 100 cm above the floor.

a) What is $v(0)$ and $s(0)$?

b) What is the velocity equation?

c) What is the maximum speed of the weight?

d) What is the displacement equation?

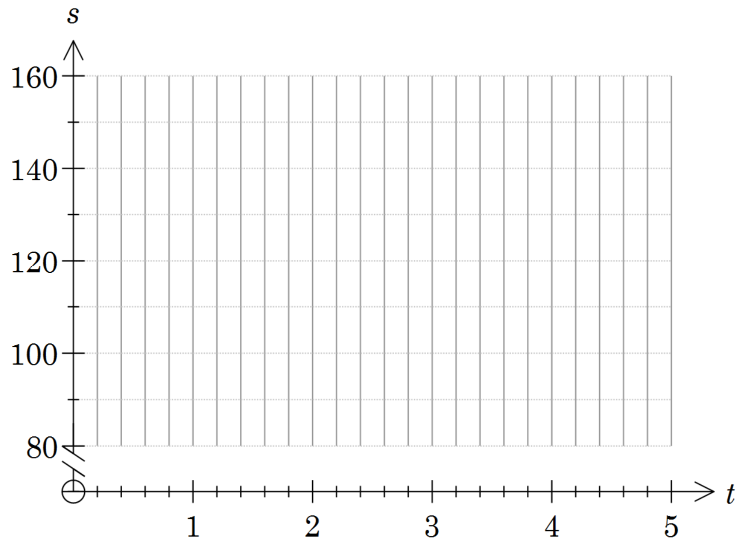
e) What is the velocity when the weight is closest to the ground?

f) Describe the motion.

g) If the initial velocity was –10 cms^{-1}

(i) Determine the new displacement function.

(ii) Draw a graph and describe the motion.



3. A model electric vehicle is accelerated along a straight track. The acceleration decreases over time as a maximum speed is reached. This is when the resistance forces balance or counteract the force generated by the motor. If the acceleration is modelled by the equation $a(t) = 360e^{-1.2t}$ cms^{-2} and the vehicle starts from rest at the origin:

a) What are the velocity and displacement functions?

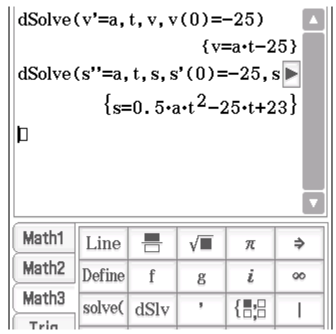
b) What is the vehicle's top speed?

c) How long does it take for the vehicle to reach 95% of its top speed? Hint: Draw a graph of the velocity function and use [Analysis | G-Solve | x-cal/y-cal | x-cal] or use solve in Main.

d) How long will it take the vehicle to reach the end of the 8 m track?

Learning notes

Newton's First law of motion can be expressed as $a = \frac{F}{m}$. So if the forces on a body of fixed mass are known then so too is the acceleration.

• Using dSolve	

Activity 18 Tax time

Aim: Determine total change given the rate.

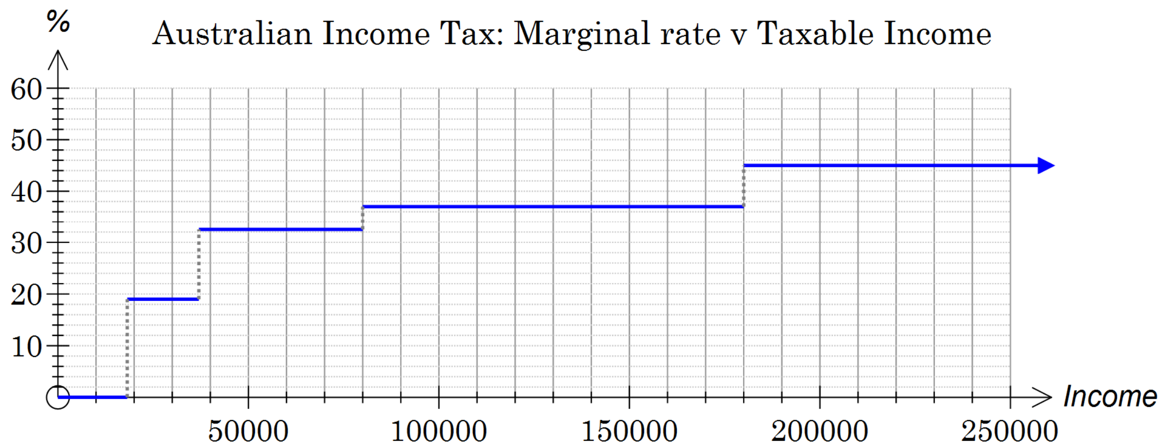
An extract from "Maths in Politics"

"Our tax system is based on the principle of fairness, those who can afford to pay more should pay more. This proposal uses a "smooth" function to calculate tax rather than the current steps. Once the threshold has been reached you pay more of every dollar you earn in tax. Vote for this for a fairer tax system."

1. The current situation:

Australian income tax scales 2014-2015

Taxable income	Tax on this income	Marginal rate
0 – $18,200	Nil	0
$18,201 – $37,000	19c for each $1 over $18,200	19%
$37,001 – $80,000	$3,572 plus 32.5c for each $1 over $37,000	32.5%
$80,001 – $180,000	$17,547 plus 37c for each $1 over $80,000	37%
$180,001 and over	$54,547 plus 45c for each $1 over $180,000	45%

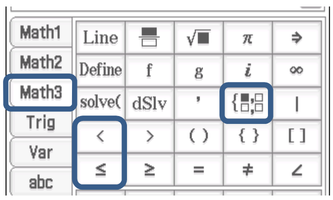


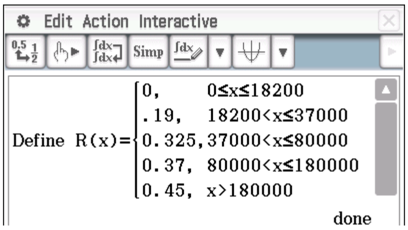
a) Shade the graph to show the amount of tax due on an income of $100 000.

b) Define a piece-wise function $R(x)$ on ClassPad to calculate the marginal tax rate on a taxable income of $\$x$ and check that the correct rates are generated.

Define function

- Open the Math3 tab on the Keyboard
- Tap [template icon] a two piece template appears, tap [template icon] again to add another piece to the template
- < and ≤ are also available from the Math3 tab
- Once the template is complete, select it all and [Edit | Copy]
- [Interactive | Define] and paste in the template result
- Edit as required

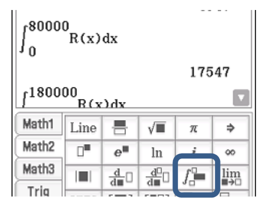




c) Calculate the tax payable on incomes of

Calculate tax

- Calculate the definite integral from 0 to the taxable income



(i) $20,000

(ii) $40,000

(iii) $100,000

(iv) $180 000

2. Model A

A new model is proposed with the rate increasing linearly from 0% at the tax free threshold $18 200 up to 45% at $100 000, i.e.

$$\text{RA}(\$x) = \begin{cases} 0 & , \quad x \le 18200 \\ \dfrac{x-18200}{100000-18200} \times 0.45 & , \quad 18200 < x \le 100000 \\ 0.45 & , \quad x > 100000 \end{cases}$$

Enter the function and check that the tax rates are correct.

a) Calculate the tax due under this system for the amounts

(i) $20,000

(ii) $40,000

(iii) $100,000

(iv) $180 000

b) Which of the incomes from part a) would pay more tax under Model A than currently (Q1)?

c) Model B
The government decides they lose too much tax under the proposal. Suppose the tax free threshold is reduced to $15 000 and the maximum rate of 45% starts at $80 000. Under this proposal which of the incomes above would pay more tax than currently (Q1)?

3. Model C
A third scenario is suggested where the rate follows a sinusoidal function.

$$\text{RC}(\$x) = \begin{cases} 0 & , \quad x \le 15000 \\ 0.225\left(1+\sin\left(\dfrac{\pi(x-45000)}{60000}\right)\right) & , \quad 15000 < x \le 75000 \\ 0.45 & , \quad x > 75000 \end{cases}$$

Enter the piecewise function.

a) Calculate the tax due under Model C for the incomes

(i) $20,000

(ii) $40,000

(iii) $100,000

(iv) $180 000

b) Which of the incomes from part a) would pay more tax under Model C than currently (Q1)?

A newspaper journalist wants information for an article on the scenarios outlined in Q2 and Q3. In particular which incomes would benefit and which would pay more.

4. For the current tax system the piece-wise tax function T(x) is shown below:

$$\text{Define } T(x)=\begin{cases} 0, & 0\le x\le 18200 \\ .19(x-18200), & 18200<x\le 37000 \\ 0.325x-8543, & 37000<x\le 80000 \\ 0.37x-12053, & 80000<x\le 18000 \\ 0.45x-26453, & x>180000 \end{cases}$$

done

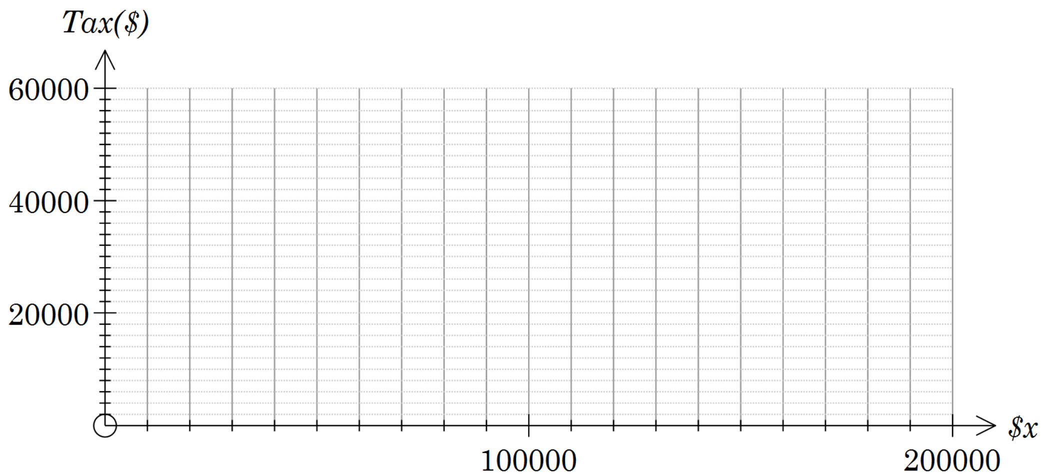
Enter the function into your ClassPad.

a) Between \$37000 and \$80000 tax due is \$3572 plus 37 cents in the dollar, i.e. $\frac{dT}{dx}=0.37$ and $T(37000)=3572$. Use integration to justify the piece of the tax function $3572+0.325\times(x-37000), 37000<x\le 80000$.

b) For Model B, Q2 c), determine the piece-wise function Tn(x).

Enter the function in ClassPad.

c) Graph the functions for Tax due currently and under Model B to determine which incomes will be worse off.



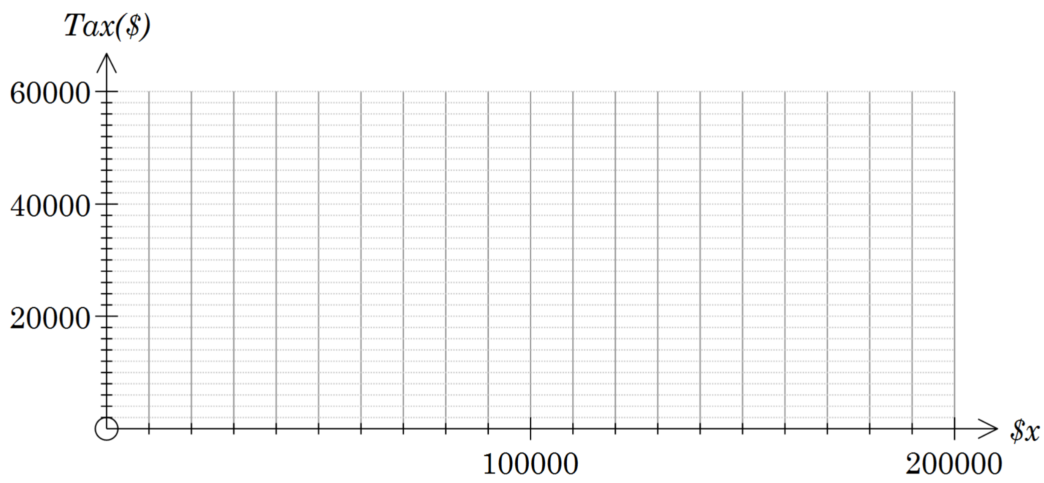
d) Determine which income has the greatest saving in tax under Model B. Hint: you could draw a graph of the difference and then locate a stationary point.

5.

a) For Model C determine the piece-wise function Tm(x) for the tax due on a taxable income of \$$x$.

Enter the function in ClassPad.

b) Graph the functions for the current tax due and Model C to determine which incomes will pay more tax under Model C.

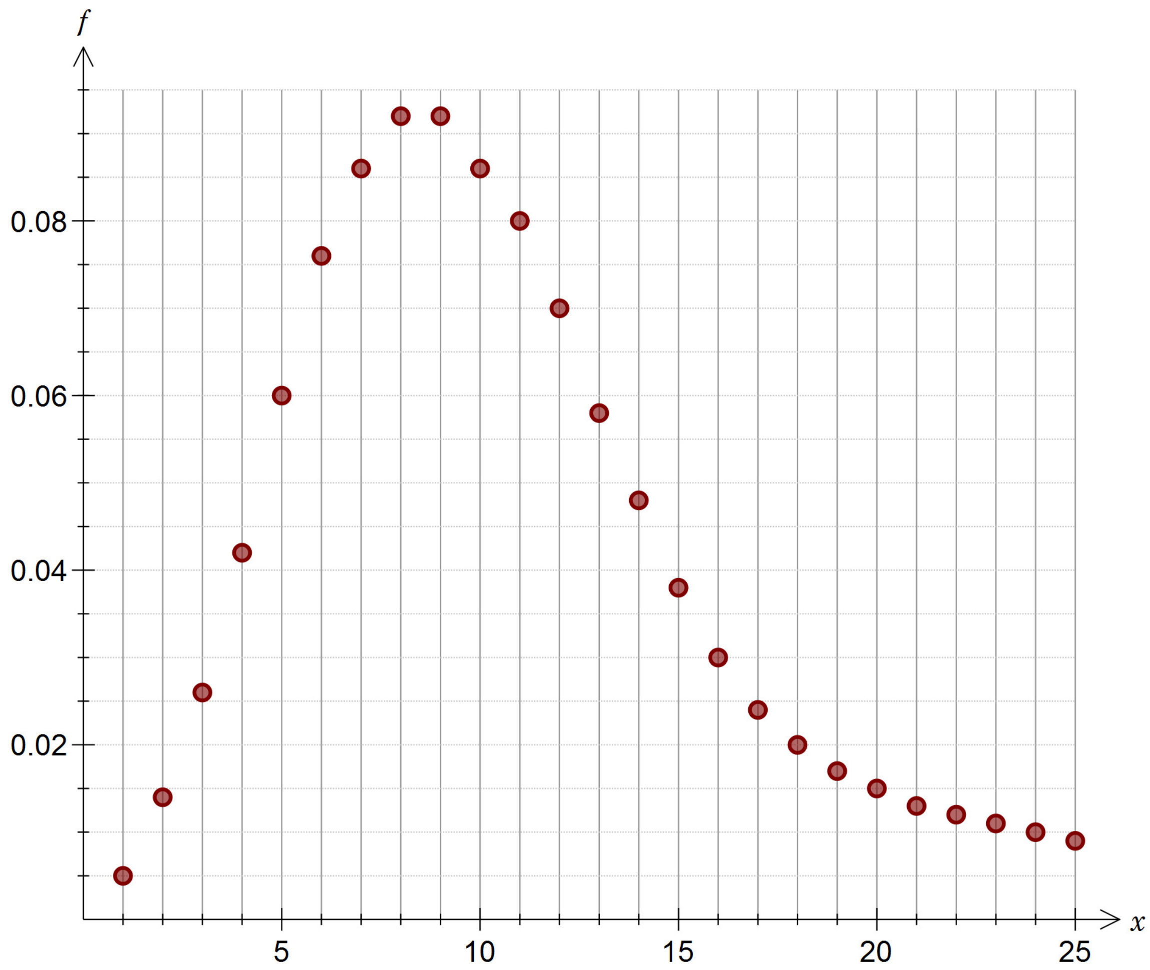


EXTENSION

6. Design a function that is a smooth curve and better fits the current tax situation.
Explain why your function is better.

Chapter 3 Discrete random variables

Activity	ClassPad applications	Key concepts
Rolling dice	Spreadsheet Statistics	Simulate dice rolling, describe the results graphically and in terms of central tendency and spread.
Up or down, the Bernoulli distribution	Spreadsheet Statistics Main	Emulate and understand the binomial probability formula.
Bernoulli trials	Statistics Main	Use the concept of Bernoulli trials and the binomial theorem to calculate number of successes and associated probabilities



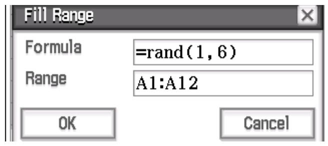
Activity 19 Rolling dice

Aim: Simulate dice rolling, describe the results graphically and in terms of central tendency and spread.

1. Simulate 12 rolls of a six-sided die.

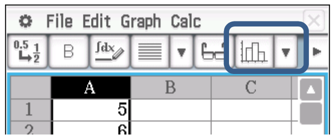
Set up Spreadsheet

- Open Spreadsheet app
- Tap in cell A1 enter the formula =rand(1,6)
- Tap in cell A1
- [Edit | Fill | Fill Range] to A12



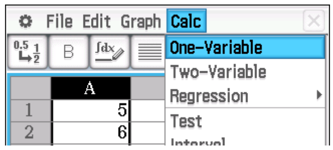
Draw the graph

- Tap in column A header
- Select from the pull-down graph menu



Calculate statistics

- Tap in the spreadsheet window
- [Calc | One-Variable] $\bar{x}$ is the mean
  Use σ_x for standard deviation



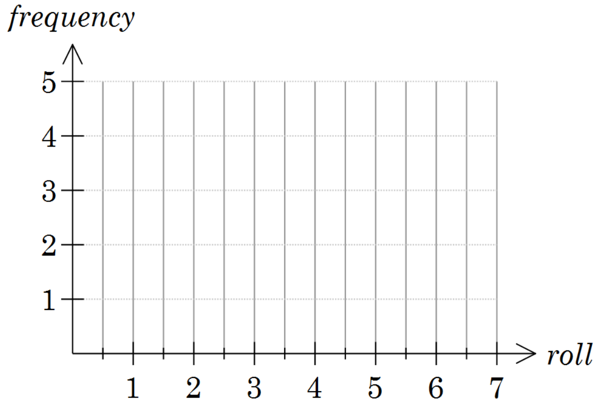
a) Draw a histogram of the results.

b) On your graph indicate

(i) the mean

(ii) the mean plus one standard deviation

(iii) the mean minus one standard deviation

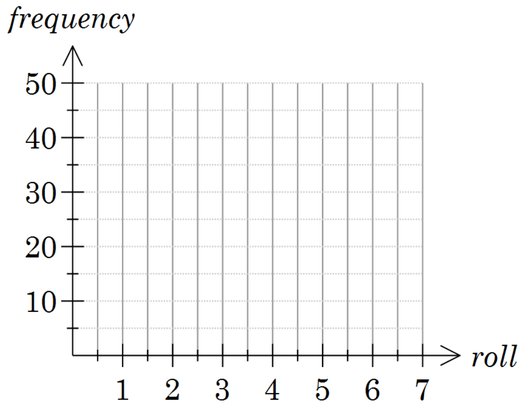


2. Modify your spreadsheet to simulate 200 rolls on a 6-sided die.

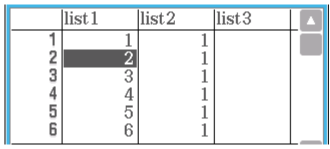
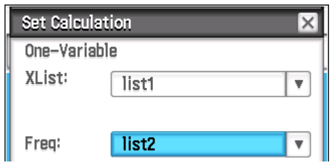
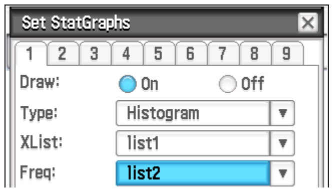
a) Draw a histogram of the results.

b) On your graph indicate
(i) the mean
(ii) the mean plus one standard deviation
(iii) the mean minus one standard deviation

c) Comment on any differences you see between Q1 and this question.



3. Explore rolling a six-sided die theoretically.
Define the Random Variable $X = \{1, 2, 3, 4, 5, 6\}$ to represent the possible outcomes from rolling a 6-sided die.

Calculate expected value • Open Statistics app • Enter the possible values i.e. 1 to 6 in list1 • Enter frequency in list 2 (Enter 1 as each outcome is equally likely)	
• [Calc | One-Variable] • Select list1 for XList list2 for Freq • Tap OK (The mean is the expected value)	
Draw the histogram • Tap  to set graph type • Make selections as shown and tap Set • Tap  • Set HStart to 0.5 and HStep to 1	

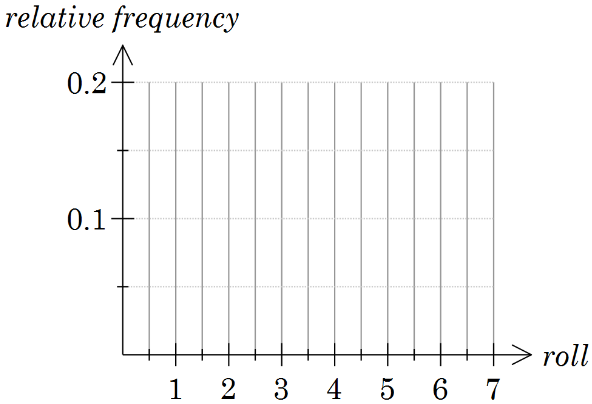
a) Draw a histogram showing the uniform distribution.

b) On your graph indicate

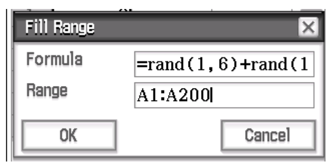
(i) the expected value or mean

(ii) the mean plus one standard deviation

(iii) the mean minus one standard deviation



4. Simulate 200 rolls of a pair of 6-sided dice with the random variable being the sum.

Modify the Spreadsheet formulae • Tap in cell A1 enter the formula =rand(1,6) + rand(1,6) • [Edit | Fill | Fill Range] A1:A200 • [File | Recalculate] to rerun the simulation	

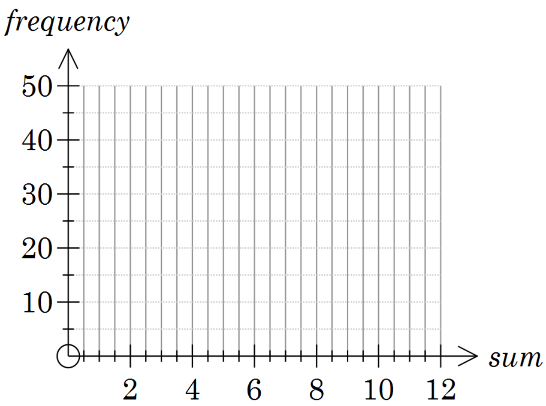
a) Draw a graph showing the distribution.

b) On your graph indicate

(i) the mean

(ii) the mean plus one standard deviation

(iii) the mean minus one standard deviation



5. Define the Random Variable $X = \{2,3,4,5,6,7,8,9,10,11,12\}$ to represent the possible sums from rolling two 6-sided die.

a) Complete the tables for the possible outcomes and their frequencies.

		Die 1					
		1	2	3	4	5	6
Die 2	1	2					
	2		4	5			
	3						
	4					9	
	5						
	6						

Sum	Frequency
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	

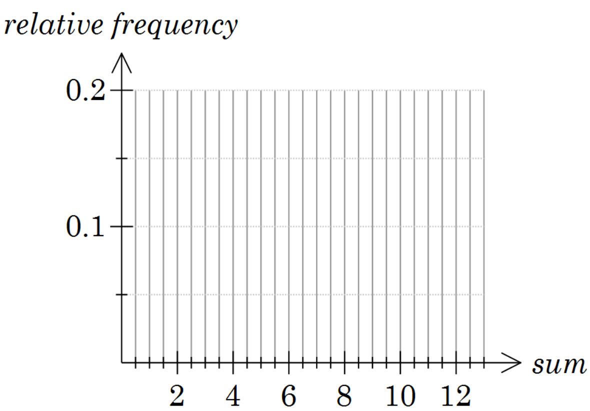
b) Draw a histogram showing the distribution.

c) On your graph indicate

(i) the expected value or mean

(ii) the mean plus one standard deviation

(iii) the mean minus one standard deviation



6. Simulate 200 rolls of a pair of dice with the random variable being the difference between the two dice.

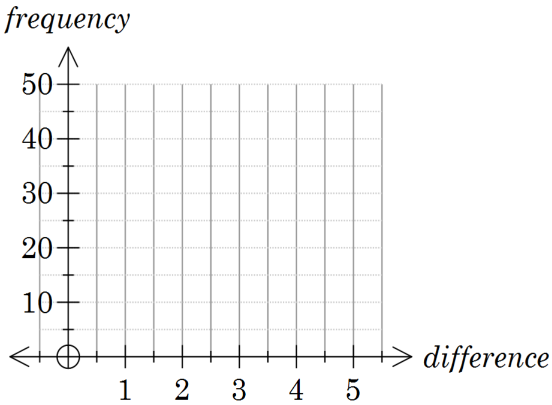
a) Draw a graph showing the distribution.

b) On your graph indicate

(i) the mean

(ii) the mean plus one standard deviation

(iii) the mean minus one standard deviation



c) Complete the tables for all possible outcomes and their frequencies.

		Die 1					
		1	2	3	4	5	6
Die 2	1	0					
	2		0	1			
	3						
	4						
	5			2			
	6						

Difference	Frequency
0	
1	
2	
3	
4	
5	

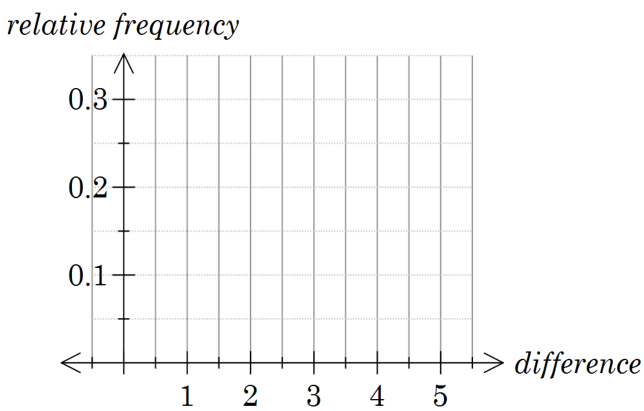
d) Draw a histogram showing the distribution.

e) On your graph indicate

(i) the expected value or mean

(ii) the mean plus one standard deviation

(iii) the mean minus one standard deviation



7. Simulate 200 rolls of a pair of dice with the random variable being the product of the two dice.

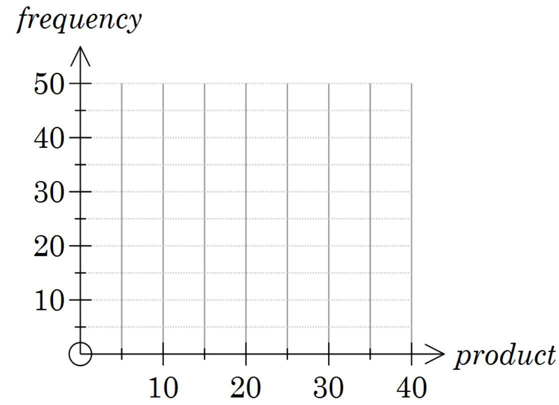
a) Draw a graph of the distribution.

b) On your graph indicate

(i) the mean

(ii) the mean plus one standard deviation

(iii) the mean minus one standard deviation



c) Complete the tables for the possible outcomes and their frequencies.

		Die 1					
		1	2	3	4	5	6
Die 2	1	1					
	2		4	6			
	3						
	4					20	
	5						
	6						

Product	Frequency
1 – 5	
6 – 10	
11 – 15	
16 – 20	
21 – 25	
26 – 30	
31 – 35	
36 – 40	

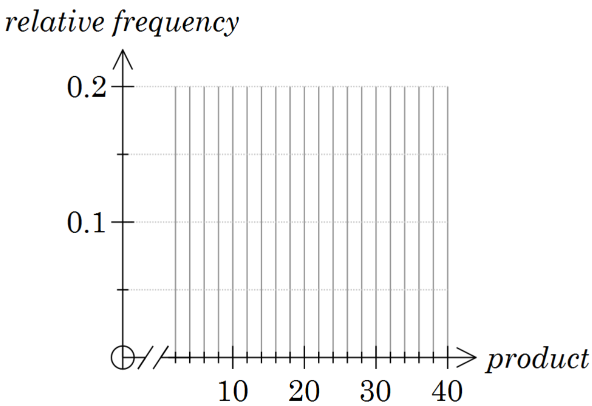
d) Draw a histogram showing the distribution.

e) On your graph indicate

(i) the expected value or mean

(ii) the mean plus one standard deviation

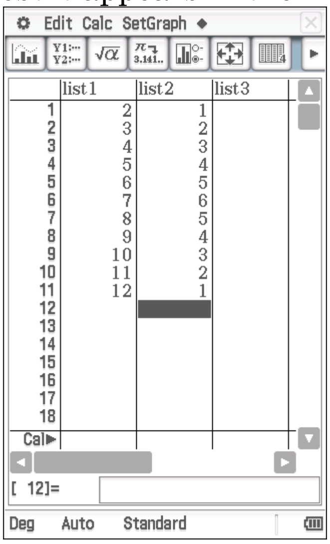
(iii) the mean minus one standard deviation

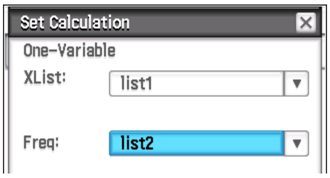


Learning notes

For each of the distributions it is desirable to estimate after drawing the graph and then do the calculations. This will help give an intuitive feel for using mean and standard deviation to describe the distribution.

Q5 For the expected value the random variable has values between 2 and 12 with frequencies that can be obtained from a 2-way table. The expected value can be calculated by putting the random variable in column A and the number of times the result appears in the 2-way table in column B .





The spreadsheet will automatically scale the histogram.
Select [Calc | Bin Width] from the graph window to alter settings.
In Statistics you can control the start and step when drawing the histogram. In Q6 ensure HStart is set to –0.5 and HStep to 1 to ensure each column is centred on the integer value and 0 is displayed.

Q7 Alter the formula to rand(1,6) × rand(1,6).
Note this distribution is not symmetric (it is skewed to the right).

Activity 20 Up or down, the Bernoulli distribution

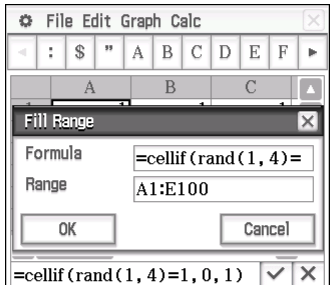
Aim: Simulate and understand the binomial probability formula.

A quarter of the cards in a pack are face up. A "hand" of 5 cards is dealt.

1. Create simulation
   The Random variable X is defined as 0 if the card is face up and 1 if the card is face down.

Set up Spreadsheet

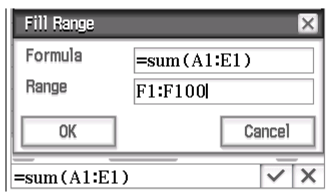
- Open Spreadsheet app
- Tap in cell A1 enter the formula =cellif(rand(1,4)=1,0,1)
  (0 represents face up and 1 represents face down)
- Tap in cell A1
- [Edit | Fill | Fill Range] to A1:E100



Calculate number of face down cards

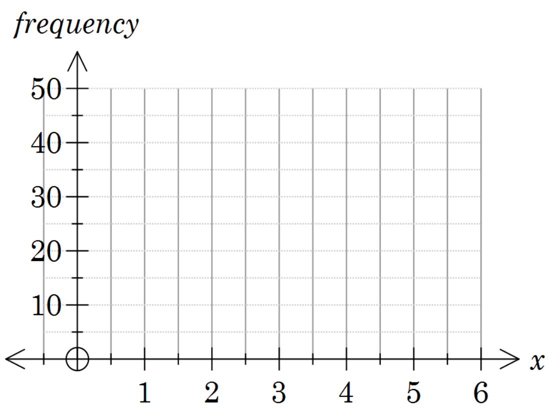
- Tap in cell F1 and enter the formula =sum(A1:E1)
- Fill the cells F1:F100 with the formula

Column F now has the number of face down cards for 100 hands



a) Draw a histogram of your results.

b) On your graph indicate
   (i) the mean
   (ii) the mean plus one standard deviation
   (iii) the mean minus one standard deviation



c) Comment on the histogram in terms of its shape, centre and spread. You may like to run the simulation more times to gain a better appreciation of the long term shape of the distribution.
(Select [File | Recalculate] from the Spreadsheet window)

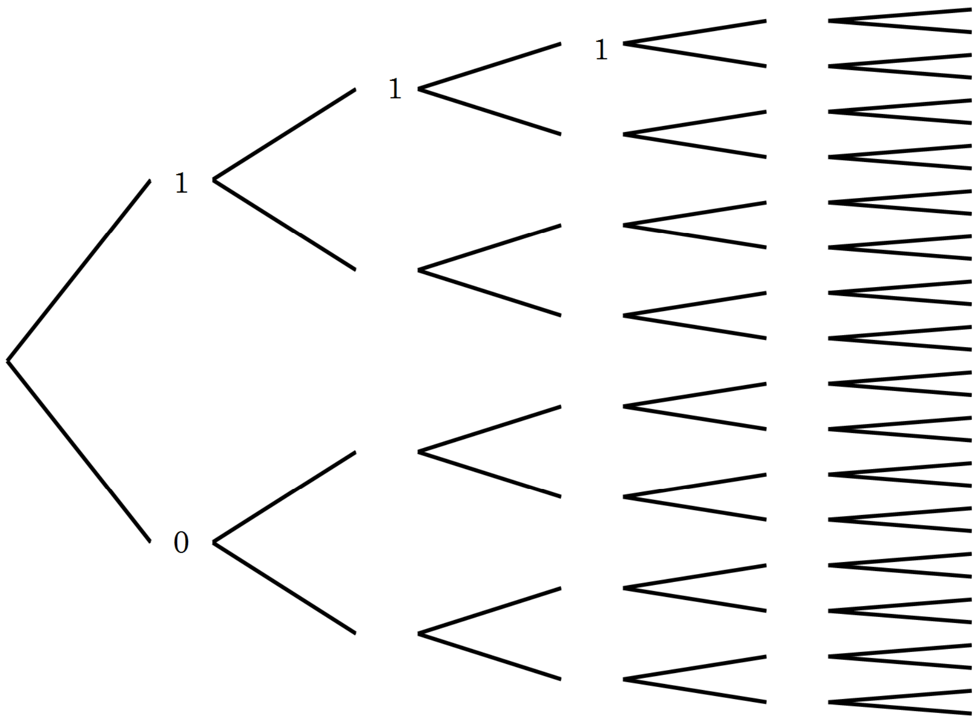
2. Explore the scenario theoretically.

a) What is the probability of getting 5 face up cards?

b) Complete the table to list all the possible arrangements of the 5 cards.

Number of face down cards	Arrangements
0	
1	{0,0,0,1,0}
2	
3	
4	
5	

c) Complete the tree diagram to show all the arrangements from b).



d) Highlight the branches with 2 face down cards.

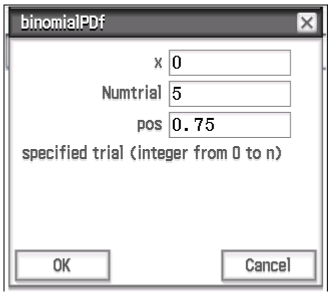
e) Summarise your results in the following table.

Number of face down cards (x)	# of arrangements from b)	Probability of a branch of x face down cards	$P(X = x)$ (3 s.f.)
0			
1			
2			
3			
4			
5			

3. Explore the binomial distribution function

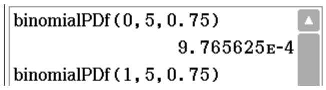
Use binomPDf function

- Open Main window
- [Interactive | Distribution/Inv.Dist | Discrete | BinomialPDf]
- The three fields (in this case) are
  x: the number of face down cards
  Numtrial: The number of cards dealt
  pos: Probability the card is face down



Alternatively

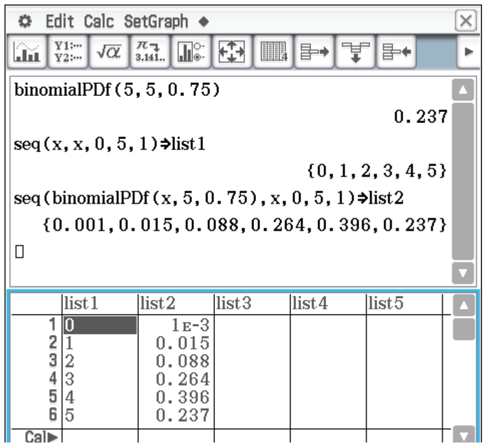
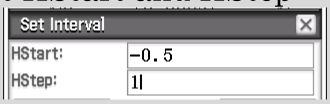
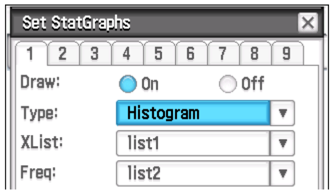
- The command can be entered directly into Main
- Highlight, drag and edit to repeat for the other possibilities.



a) Complete the table

x	0	1	2	3	4	5
binomPDf(x,5,0.75)						

b) Compare your results in Q2 e) and Q3 a). What do you notice?

Generate results as a sequence and store in Statistics lists • In Main • [Action | List | Create | seq] • Complete the entries to generate the outcomes 0 to 5 • Store in list1 • Store the probabilities in list 2. See statement in screenshot. Note Number format has been set to Fix3 for tidiness.	
Calculate statistics • Open Statistics app • [Calc | One-Variable] using list1 for XList and list2 for frequency	
Draw histogram • Tap to Set graph type • Tap to graph • Set HStart and HStep 	

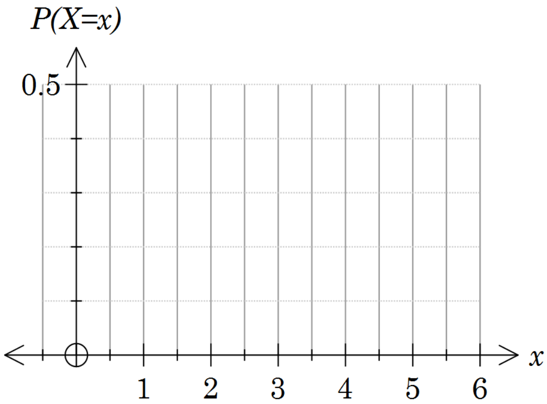
c) Draw a histogram of your results.

d) On your graph indicate

(i) the mean

(ii) the mean plus one standard deviation

(iii) the mean minus one standard deviation



Learning notes

It is assumed that the cards are replaced after each card is dealt. This is equivalent to considering a deck with a very large number of cards.

Activity 21	Bernoulli trials

Aim: Use the concept of Bernoulli trials and the binomial theorem to calculate number of successes and associated probabilities

James Randi is a magician and sceptic who believes those who claim to have paranormal powers are deluded or fraudsters. He has set up a foundation that has a $1 million prize to anyone who can prove they possess such powers. http://www.randi.org/site/index.php/1m-challenge.html.

Max claims to be able to divine water. How might a test be designed that would enable Max to demonstrate his claim?

It is proposed that four pipes are buried in a field and water can flow through them independently.

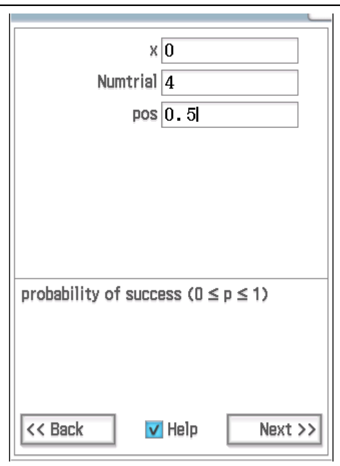
1. Complete the table for the different arrangements of water flowing or not flowing in each pipe.

Number of pipes carrying water.	0	1	2	3	4
List the different arrangement of pipes carrying water.	XXXX	XXXO XXOX XOXX OXXX			
The probability of this number of pipes carrying water. (Assume 50% chance of water flowing in each pipr					

2. a) Duplicate the results from Q1.

Calculate value for a Bernoulli trial

- Open Statistics app
- Select [Calc | Distribution]
- Select Binomial PD
- Check the Help box to get descriptions of the distribution or parameter
- Tap Next >>
- Complete the entries.
- Tap Next >>

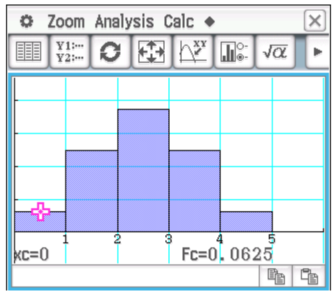


Display the histogram

- Tap [icon]

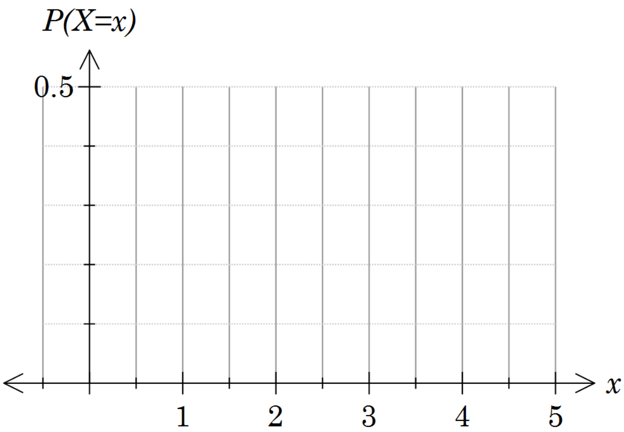
Read off values

- Select [Analysis | Trace]
- Move left and right using the arrow keys



b) Repeat with 60% probability of water flowing in any one pipe. Record the histogram and complete the table of values.

(i)



(ii)

Number of pipes with water (x)	Probability
0	
1	
2	
3	
4	

c) Use your table to determine the probability of

(i) at least two pipes carrying water.

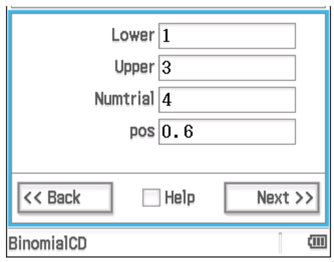
(ii) three or fewer pipes carrying water.

(iii) between 1 and 3 pipes (inclusive) carrying water.

d) Duplicate your answers to part c) using Binomial CD.

Use binomialCD

- Tap <<Back twice
  Or go back into Statistics app
- Select Binomial CD
- Tap Next >>
- Complete entries as required.



3. Describe the difference between binomialPD (Probability Distribution function) and binomialCD (Cumulative Distribution function).

4. Max approaches Randi with the following proposal.

Four pipes will be laid underground and water flows in at least two of them at a time. Max will traverse the field and if he can correctly identify the pipes carrying water that is a success. Max can proceed to the formal evaluation stage if he is able to demonstrate success on at least half of the trials.

a) Show that Max's chance of success based upon random guessing for a single trial is $\frac{1}{11}$.

b) Determine the probability that Max proceeds to the formal evaluation stage if he guesses randomly and he has

(i) 4 trials

(ii) 6 trials

(iii) 9 trials

5. The foundation comes back with a counter proposal.

Five pipes will be laid and water allowed to flow in at least two of them at a time. If Max can correctly identify the pipes carrying water that counts as a success. If Max is able to demonstrate success on at least half of the eight different trials the Foundation will accept Max into the formal evaluation stage.

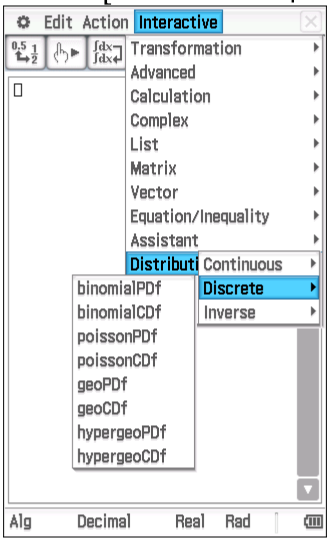
a) What is Max's chance of success based upon random guessing for a single trial?

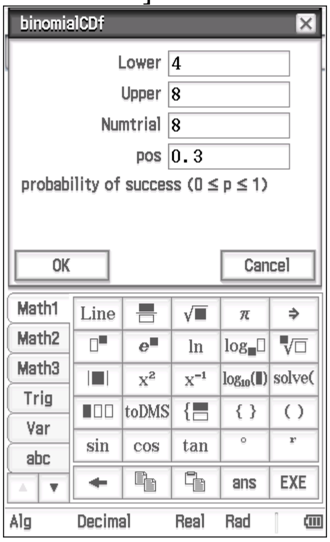
b) What is Max's chance of being accepted into the formal evaluation stage assuming he guesses randomly on each of the eight trials?

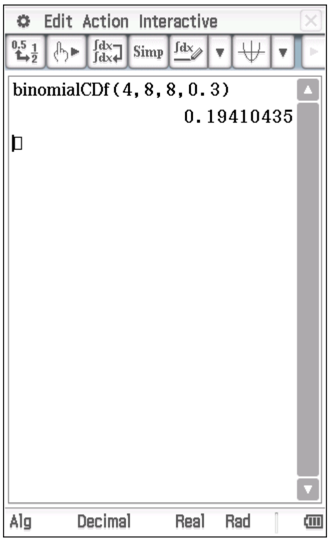
c) The foundation decides to set a threshold probability of 0.0001 (they have received thousands of claims) before allowing a claim to proceed to formal evaluation. What is the least number of five pipe trials they should set such that the probability of correctly guessing on half of the trials is no more than 0.0001?

Learning Notes

Distributions can also be accessed in Main.
Select [Interactive | Distribution]



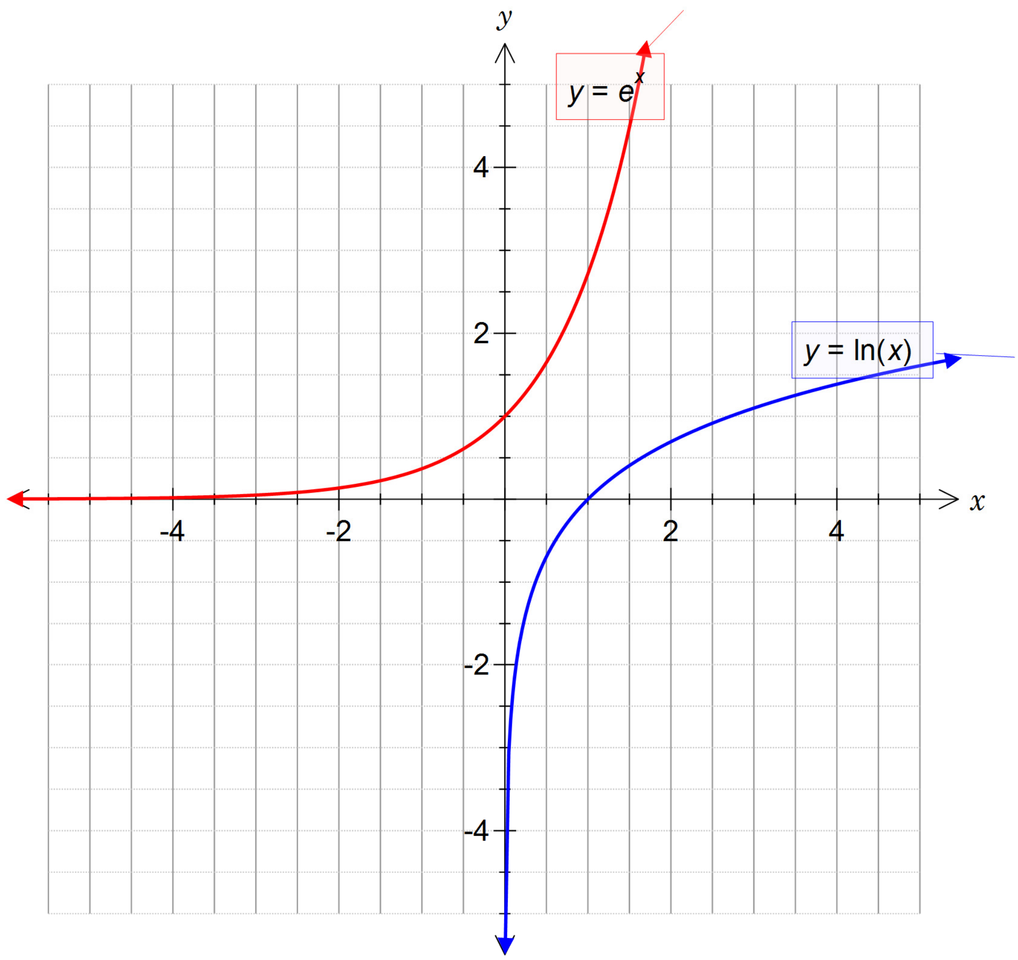




The command line can be copied and edited for further calculations.

Chapter 4 Logarithms

Investigation	ClassPad applications	Key concepts
What is log?	Main	Understand the logarithm as the index of a number expressed as a power
Log laws	Main	Deduce the log laws
Growth of the WWW	Statistics	Use a log scale to graph data with widely varying values
Key features of logarithmic functions	Graph&Table	Determine asymptotes and intercepts of logarithmic functions
Applications of logs	Main	Solve problems using logs
Derivative of ln(x)	Graph&Table	Appreciate the derivative of $y = \ln x$
Slope fields	DiffEqGraph	Explore slope fields
Integrate 1/x	DiffEqGraph	Integrate 1/x



Activity 22 What is log?

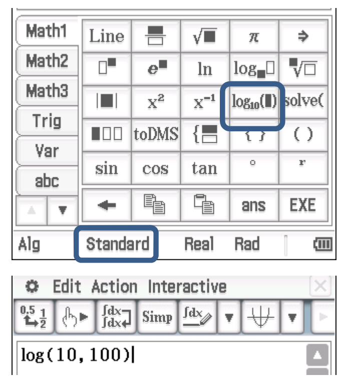
Aim: Understand the logarithm as the index of a number expressed as a power

1. Log base 10

a) Complete the table.

Use ClassPad to evaluate $\log x$

- In Main
- Ensure you are in Standard mode
- From Keyboard Math1 tab, tap $\log_{10}(\blacksquare)$
- Enter the number
- Press EXE



Number (x)	x as a power of 10	$\log x$
10		
10000		
1		
	10^{-2}	
0.001		
$\frac{1}{10}$		
	$10^{\frac{1}{2}}$	
$\left(\sqrt{10}\right)^3$		

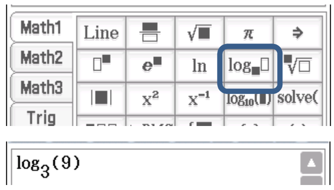
b) Describe what the log of a number means.
Use the examples from part a) to help.

2. Logs of other bases

a) Complete the table

Use ClassPad to evaluate $\log_b x$

- From Keyboard Math1 tab, tap $\log_{\blacksquare}\square$
- Enter the base and the number



Number (x)	Base (b)	x written as a power of the base	$\log_b x$
	3	3^2	
81	3		
1024	2		
	13	13^4	
125	5		
$\sqrt{5}$	5		
	2		4
	4		1.5
	3.7	$3.7^{1.24}$	

b) Describe the meaning of $\log_b x$

c) What is the value of

(i) $\log 10^a$

(ii) $\log_b b^x$

d) If $a = b^x$ then what is $\log_b a$?

Activity 23 Log laws

Aim: Deduce the log laws

1. Ensure ClassPad is in Standard mode
   a) Record the ClassPad output for the following expressions.
      (i) $\log(15)$
      (ii) $\log(26)$
      (iii) $\log(35)$
      (iv) $\log(77)$
   b) Use part a) to suggest an alternative expression for $\log(a \times b)$

2.
   a) Record the ClassPad output for the following expressions.
      (i) $\log\left(\frac{3}{2}\right)$
      (ii) $\log\left(\frac{11}{7}\right)$
      (iii) $\log\left(\frac{5}{13}\right)$
      (iv) $\log(1.4)$
   b) Use part a) to suggest an alternative expression for $\log\left(\frac{a}{b}\right)$

3.
   a) Record the ClassPad output for the following expressions.
      (i) $\log(49)$
      (ii) $\log(125)$
      (iii) $\log\left(\frac{1}{9}\right)$
      (iv) $\log(7^6)$
      (v) $\log(3^a)$

b) Use part a) to suggest an alternative expression for $\log\left(a^x\right)$

4. Predict alternative expressions for the following and then check with ClassPad

a) $\log 105$

b) $\log 154$

c) $\log(3\times5\times7\times11)$

d) $\log 18$

e) $\log\frac{3}{8}$

f) $\log\frac{256}{81}$

EXTENSION

Verify that the rules you have induced apply to logs with other bases including $\log_e$ (often written ln).

Learning notes

In this activity we are using CAS to generate alternative ways of expressing some logarithms.

These are the three log laws:

$\log a + \log b = \log(ab)$

$\log a - \log b = \log\frac{a}{b}$

$\log a^n = n\log a$

Activity 24 Growth of the WWW

Aim: Use a log scale to graph data with widely varying values

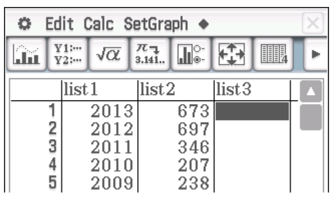
This table shows the number of websites on the web (WWW) in June each year.

Year (June)	Websites	Change	Internet Users	Websites launched
2013	**672,985,183**	-3%	2,756,198,420	
2012	**697,089,489**	101%	2,518,453,530	
2011	**346,004,403**	67%	2,282,955,130	
2010	**206,956,723**	-13%	2,045,865,660	Pinterest
2009	**238,027,855**	38%	1,766,206,240	
2008	**172,338,726**	41%	1,571,601,630	Dropbox
2007	**121,892,559**	43%	1,373,327,790	Tumblr
2006	**85,507,314**	32%	1,160,335,280	Twttr
2005	**64,780,617**	26%	1,027,580,990	YouTube, Reddit
2004	**51,611,646**	26%	910,060,180	Thefacebook, Flickr
2003	**40,912,332**	6%	778,555,680	WordPress, LinkedIn
2002	**38,760,373**	32%	662,663,600	
2001	**29,254,370**	71%	500,609,240	Wikipedia
2000	**17,087,182**	438%	413,425,190	Baidu
1999	**3,177,453**	32%	280,866,670	PayPal
1998	**2,410,067**	116%	188,023,930	Google
1997	**1,117,255**	334%	120,758,310	Yandex
1996	**257,601**	996%	77,433,860	

Source: http://www.internetlivestats.com/total-number-of-websites/ Feb 2014

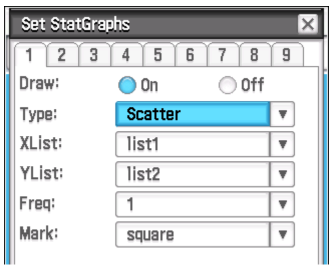
Enter data in Statistics

- Open Statistics app
- Enter the year in list1
- Enter the number of websites (in millions) in list2

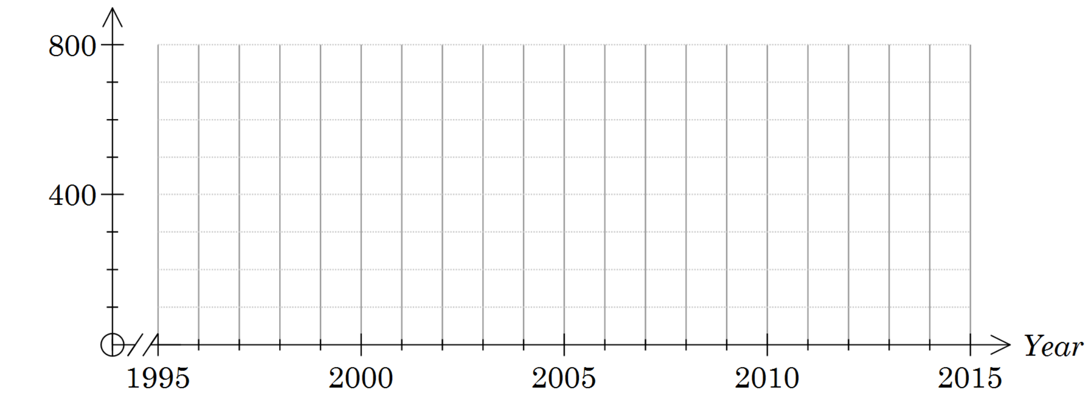


Set and draw the graph

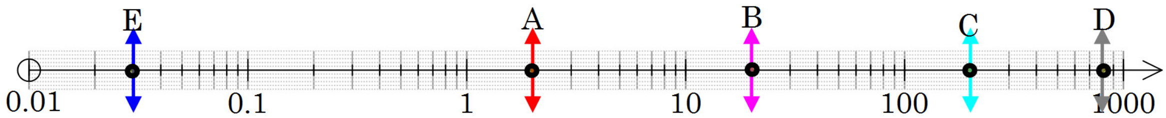
- Tap
- Ensure StatGraph1 is set as shown
- Tap to draw the graph



1. Draw the graph



2. This is a log scale. E is a point plotted at (0.03,0)

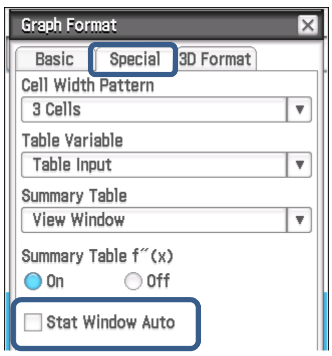
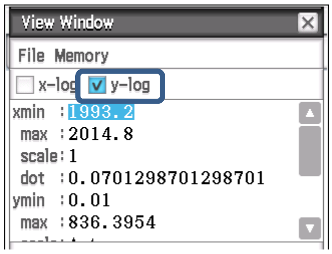


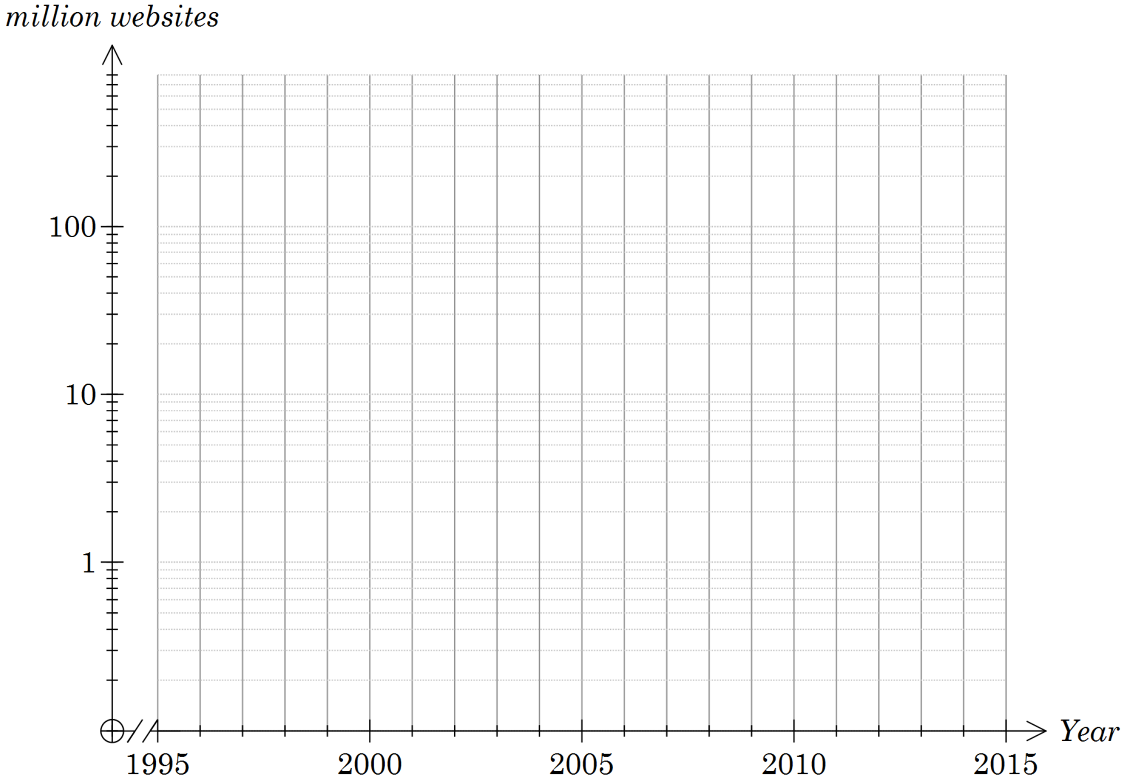
a) What are the values of points A to D?

b) Plot the points F (0.5,0), G(75,0) and H(250,0) on the scale above.

3. Record the graph with a log scale for the number of websites.

Change the vertical scale to a log scale.

<table>
<tr><td>Change to log scale for number of websites
Tap ⚙
Select Graph Format
Go to Special tab
Uncheck Stat Window Auto
Tap Set
Tap to draw the graph
Turn Stat Window Auto back on when you have finished the activity</td><td>
</td></tr>
<tr><td>Check view window
Tap
Check the values in the view window for xmin, xmax, ymin and ymax are similar to those shown
Check y-log
Tap OK</td><td>
</td></tr>
<tr><td>Tap to draw the graph</td><td></td></tr>
</table>



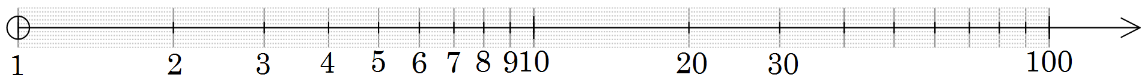
4. Why might we want to use the graph in Q3 rather than the graph from Q1?

Learning notes

In sketching the graphs an accurate plot is not required. It is sufficient to be showing the data points as a trend.

A graph with one axis using a log scale is called a semi-logarithmic graph. These are useful for showing trends in situations where growth is exponential.

Reading a log scale: this is a log base 10 scale



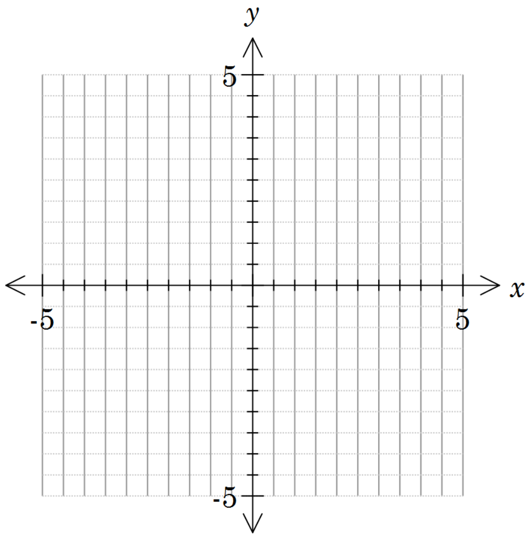
The vertical lines represent the integers 1 to 10. After 10 the next vertical line is 20 and so on. The distance between the vertical lines decreases until the next power of 10 is reached.

Activity 25 Key features of logarithmic functions

Aim: Determine asymptotes and intercepts of logarithmic functions.

For each function sketch the graph and record the asymptote and intercept(s).

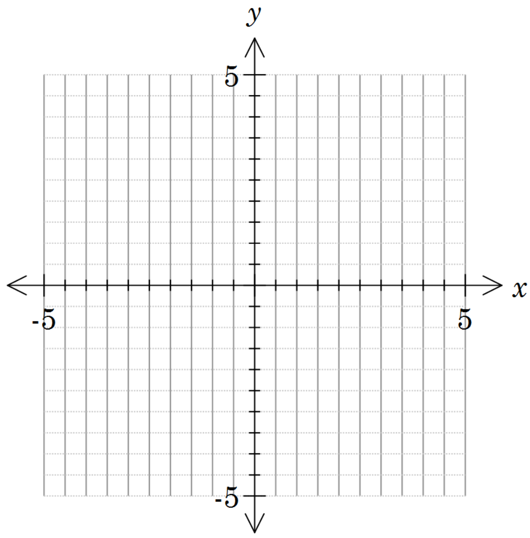
1. $y = 2^x$



Asymptote:

Intercept(s):

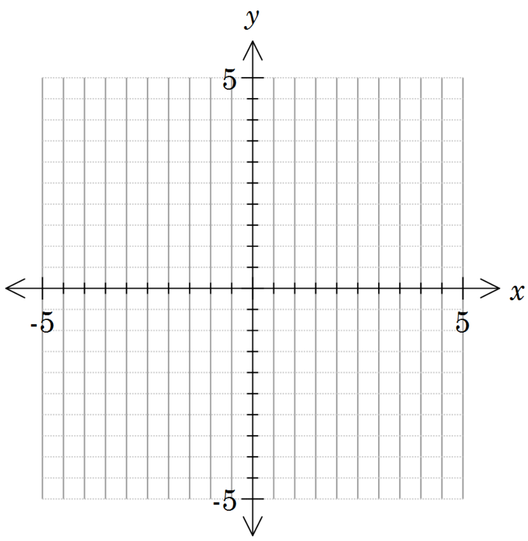
2. $y = \log_2(x)$



Asymptote:

Intercept(s):

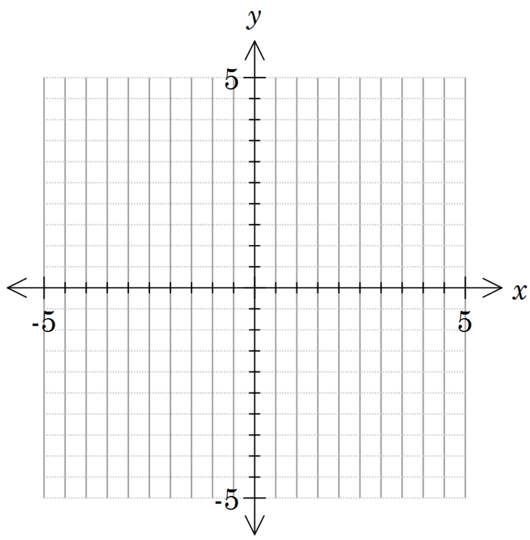
3. $y = 2^x - 2$



Asymptote:

Intercept(s):

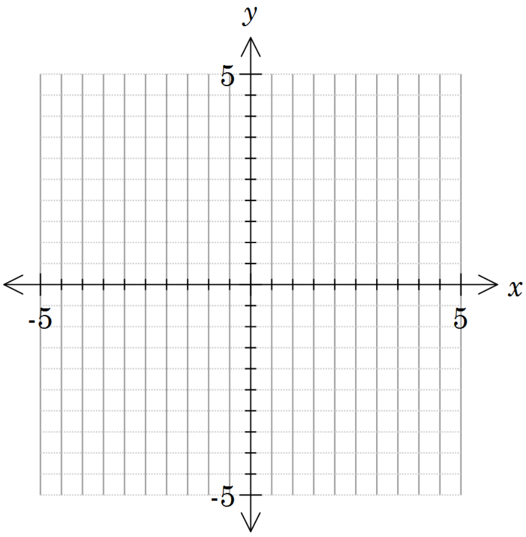
4. $y = \log_2(x+2)$



Asymptote:

Intercept(s):

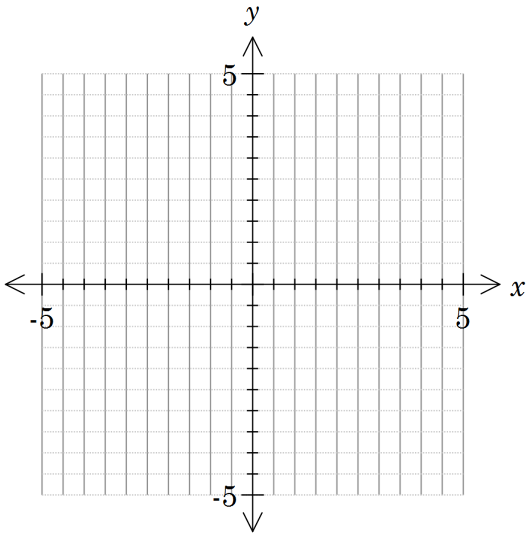
5. $y = 2^{x+1}$



Asymptote:

Intercept(s):

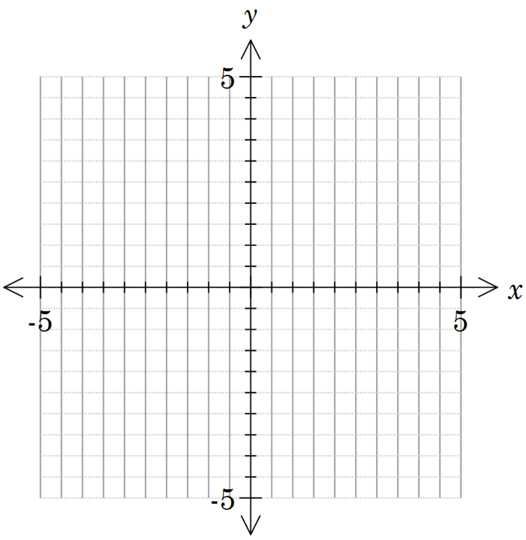
6. $y = \log_2 x - 1$



Asymptote:

Intercept(s):

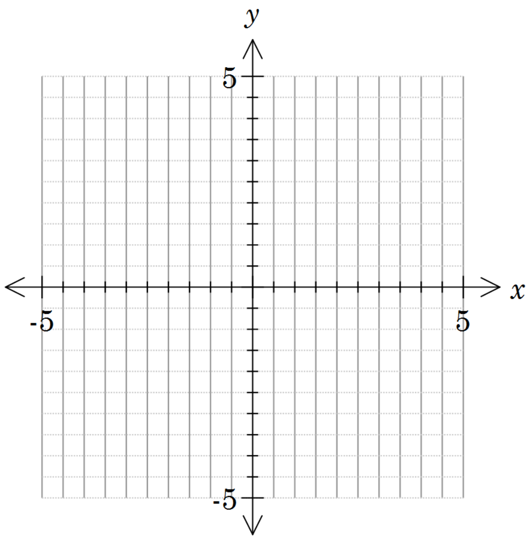
7. $y = e^{-x} - 4$



Asymptote:

Intercept(s):

8. $y = -\log_e (x + 4)$



Asymptote:

Intercept(s):

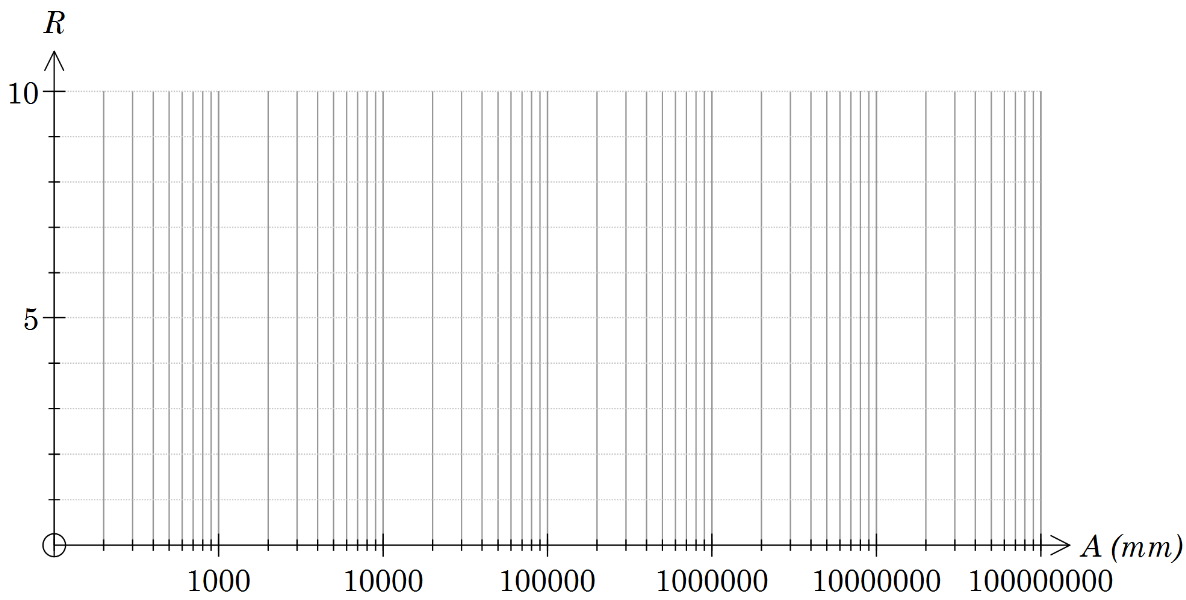
Activity 26 Applications of logs

Aim: Solve problems using logs

1. The Richter scale measure R, of the magnitude of an earthquake is given by the formula: $R = \log \frac{A}{A_0}$ where

   A is the measure of the amplitude of the earthquake wave
   A_0 is the amplitude of the smallest detectable wave (or standard wave)
   (for this activity assume A_0 is 0.05 mm).

   a) Draw a graph R v A (Note the logarithmic scale on the horizontal axis)



   b) Complete the table and plot the earthquakes on the graph

Earthquake	Year	R	*A*
Lisbon, Portugal	1755	8.5 - 9	
Valdivia, Chile	1960	9.5	
Meckering, WA	1969	6.8	
Tangshan, China	1976	7.8	
Newcastle, NSW	1989	5.6	
Christchurch, NZ	2011	6.3	
Melbourne, Vic	2014	3.2	

2. Sound is measured in a logarithmic scale using a unit called a decibel (dB). The formula looks similar to the Richter scale:

$$dB = 10\log\left(\frac{P}{P_0}\right) \quad \text{or for the Bel: } B = \log\left(\frac{P}{P_0}\right)$$

where P is the power or intensity of the sound, and P_0 is the weakest sound that the human ear can hear.

a) Show that $dB_2 - dB_1 = 10\log\left(\frac{P_2}{P_1}\right)$

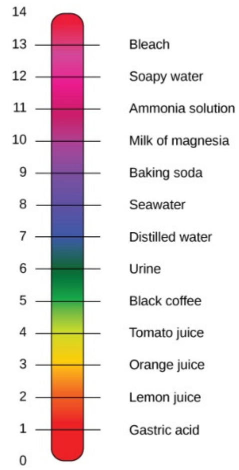
b) If the intensity doubles, what is the difference in the decibel rating?

c) If the decibel increases by 10, by what factor does the intensity increase?

d) One hot water pump has a noise rating of 50 decibels. A washing machine has a noise rating of 62 decibels. The washing machine noise is how many times as intense as the hot water pump noise?

3. Acidity is measured on the pH scale.

$\text{pH} = -\log([\text{H}])$ where [H] is the concentration of hydrogen ions in moles/litre (M).



a) What is the pH if the concentration of hydrogen ions is $3.6\times10^{-2}\,\text{M}$?

b) What is the concentration of hydrogen ions for a solution of pH 10.3?

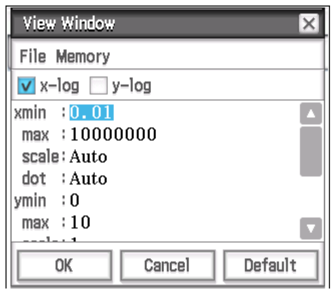
c) What is the ratio in the concentration of hydrogen ions between solutions of pH=5 and pH=4.3?

Learning notes

Q1 a)

- Enter the function in Graph&Table
- Draw the graph
- Tap [icon]
- Adjust the scales of the View Window to match the graph and set the *x*-axis to a log scale

- You can also use Trace to locate the earthquakes in b) on the graph.



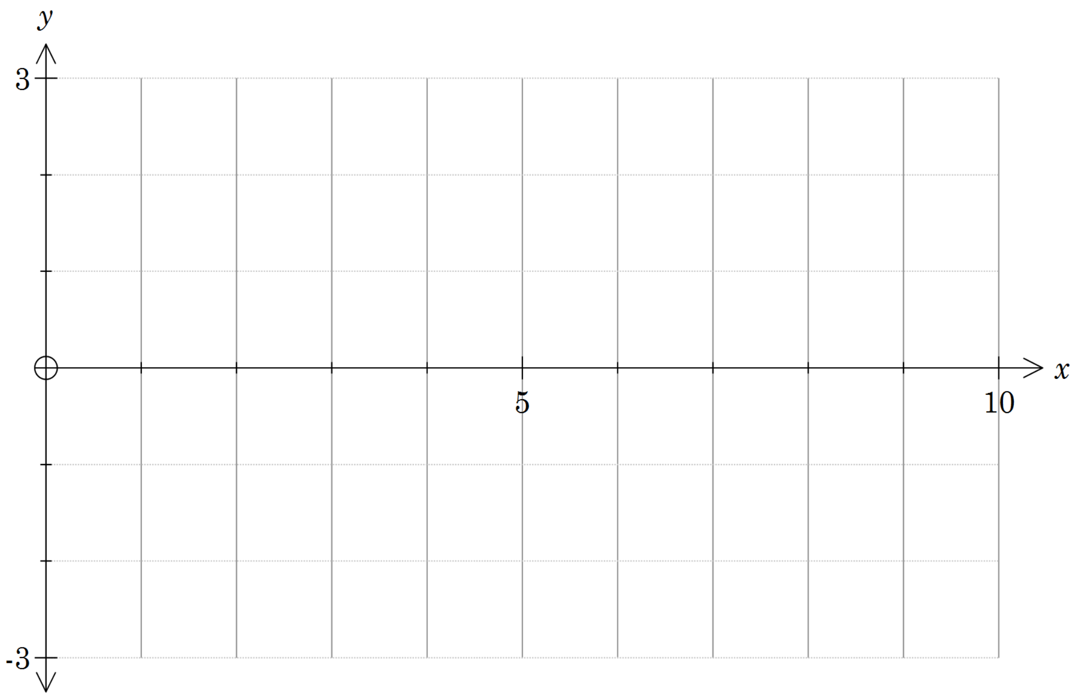
Q2 A decibel is one tenth of a Bel.

The relationship $dB_2 - dB_1 = 10\log\left(\frac{P_2}{P_1}\right)$ makes it easier to compare the ratio in intensity with the difference in the decibel measure.

Activity 27 Derivative of ln(x)

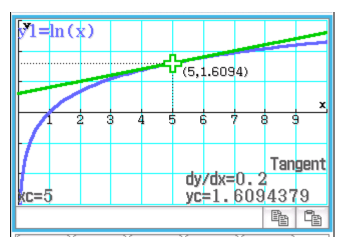
Aim: Appreciate the derivative of $y = \ln x$

1. Draw the graph of $y = \ln(x)$ on your ClassPad and copy to the grid below.



2. On your graph draw in tangents at $x = 1$ and $x = 4$.
3. Estimate the gradient of the curve at $x = 1$ and $x = 4$.
4. Complete the table

Draw tangent
- Select [Analysis | Sketch | Tangent]
- Move along the curve using the cursor controls



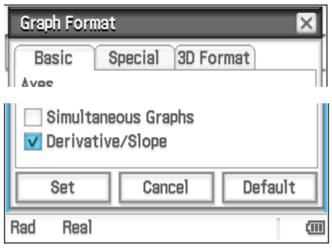
x	0.5	1	2	3	4	5
y						
$\frac{dy}{dx}$						

5. What relationship can you see between x and $\frac{dy}{dx}$?

6. Display a table of values for the derivative using ClassPad.

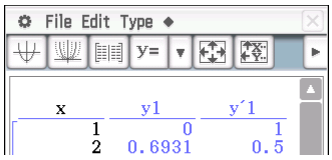
Set up

- Tap ⚙ to review/change settings
- Select Graph Format
- Tick Derivative/Slope
- Tap Set



In the define function window

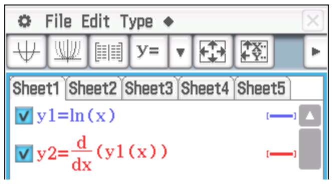
- Tap [table icon] to display a table of values
- Tap [table settings icon] to adjust the x-values displayed



Suggest a function to describe the derivative of $y = \ln(x)$.

7. Add the derivative function to your graph in Q1.

- From the Keyboard Math2 tab
- Tap [d/dx icon] and enter function as shown



Describe key features of this graph.

Activity 28 Slope fields

Aim: Explore slope fields

This example shows the direction of force around a bar magnet.

A slope field visually describes the gradient at any point in space.

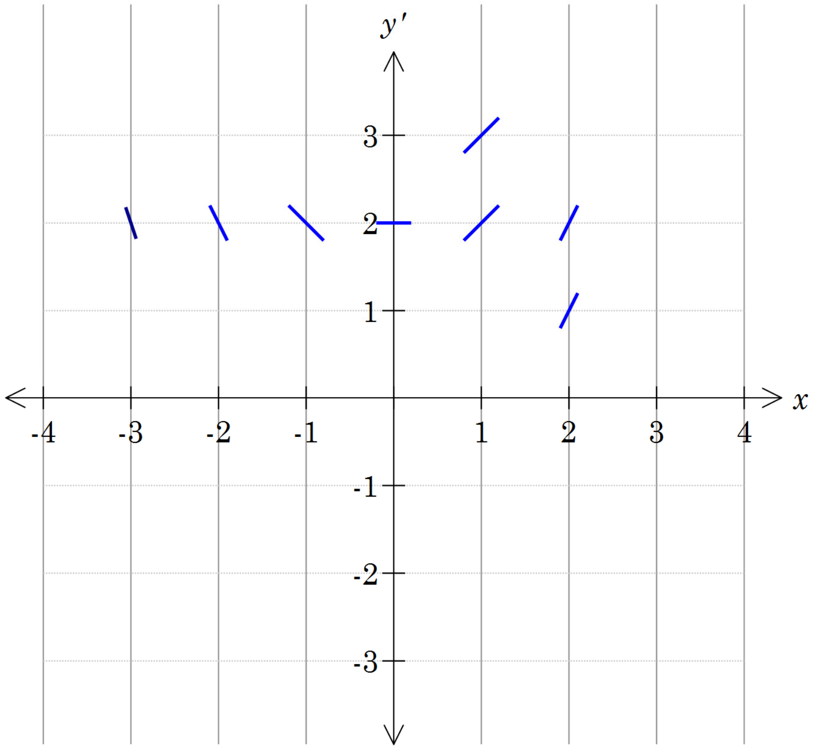
Consider the equation $\frac{dy}{dx} = x$

1. Complete a table of values

x	–3	–1	0	1	2	3
$\frac{dy}{dx}$		–1				

2.

a) On the graph below complete the slope field by drawing short line segments with the approximate gradient calculated above.



b) Draw a graph that follows the slope field and passes through

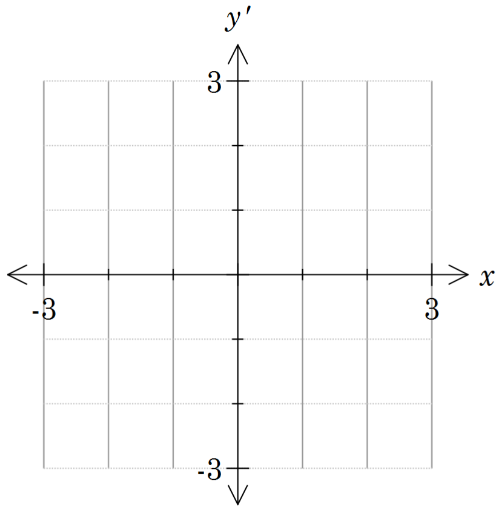
(i) (2, 0)

(ii) (–2, 3)

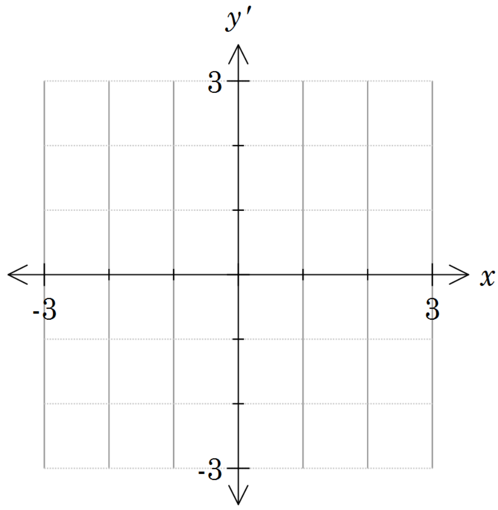
c) What are the equations of the graphs drawn in b)?

3. Use ClassPad to draw the slope fields and curve for each equation and the specified point. (See Learning notes) Copy the results to the grids below.

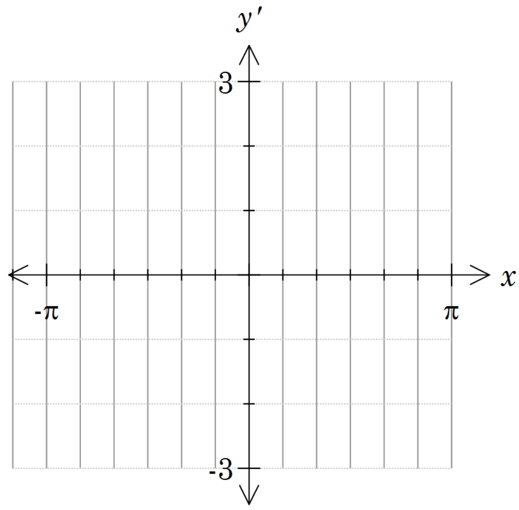
a) $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x+3}}$
through (–2, 0)



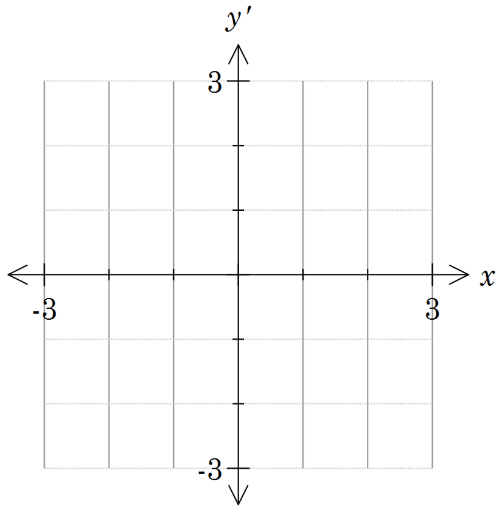
b) $\dfrac{dy}{dx} = \dfrac{1}{6}x^2 - \dfrac{x}{3}$
through (–2, 0)



c) $\dfrac{dy}{dx} = 2\sin x$
through (0, –1)



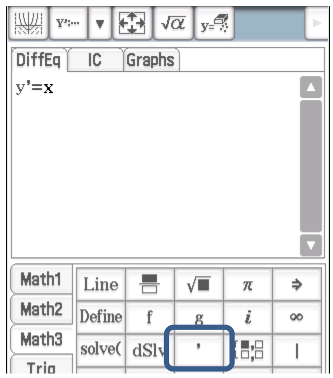
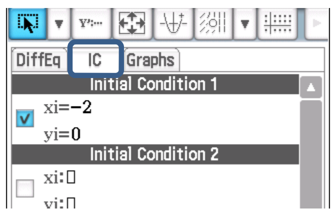
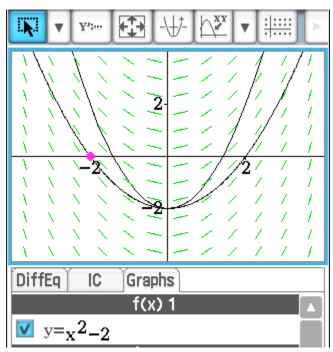
d) $\dfrac{dy}{dx} = 1 - e^{-x}$
through (0, –2)



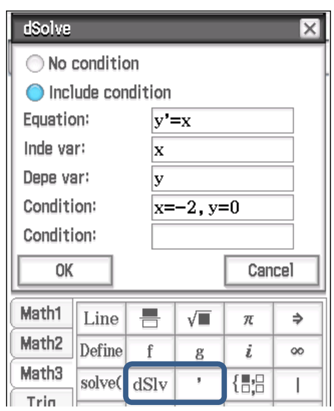
4. Determine the solution to the differential equations in Q3, i.e. state the equation of each curve.

Learning notes

Use ClassPad to draw slope fields

Open Differential Equation Grapher • Open DiffEq-Graph app	
Enter the expression for the gradient • In the DiffEq tab • Enter expression Use the Math3 tab of the Keyboard to enter ’ • Tap to draw the slope field	
Enter initial condition • Tap on IC tab • Set initial point as specified • Tap	
Use trial and error to match the curve • Click on Graphs tab • Enter equation • Tap When it is the correct equation it will overlay the curve. If not a second curve will be drawn. Use that information to refine your guess.	

Q4 A better approach is to integrate the expressions.
You can also use ClassPad’s differential equation solver function, dSolve.

Solve differential equation • [Interactive | Advanced | dSolve] • Tap include condition (given point) • Fill in the fields • Tap OK	

Activity 29 Integral of $1/x$

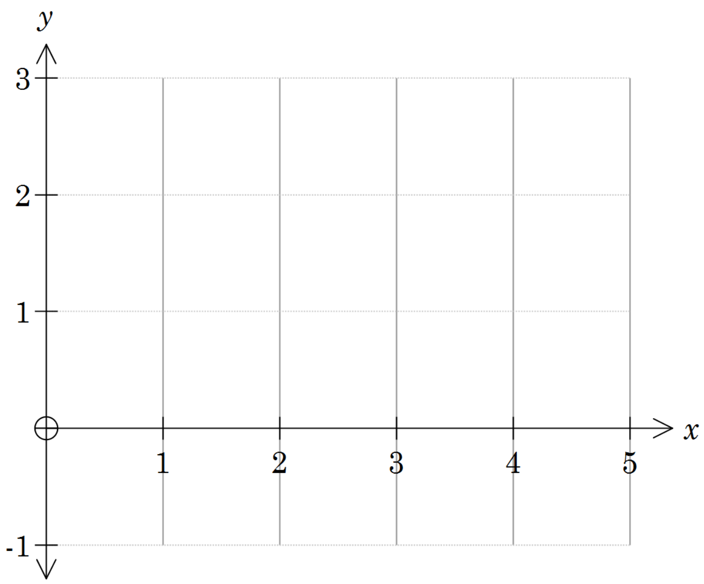
Aim: Integrate $1/x$

If the gradient is $\frac{1}{x}$ what might the original function be?

1. Complete a table of values

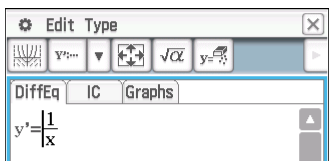
x	0.5	1	2	3	4	5
$\frac{1}{x}$		1				

2. On the graph below draw short line segments with the approximate gradient calculated above. I.e. when $x = 1$ draw a short line segment of gradient 1. This is sometimes called a slope field.



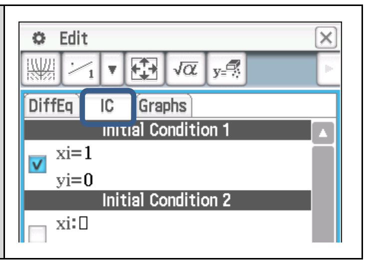
What might the graph look like if we started at the point (1,0) and tried to follow the gradient?
Try this on your graph.

3. Use ClassPad to draw the slope field

Open Differential Equation Grapher • From the Menu tap DiffEq-Graph	
Enter the expression for the gradient • In the DiffEq tab • Enter expression • Tap to draw the slope field	

Enter initial condition

- Tap on IC tab
- Set initial point to (1, 0)
- Tap

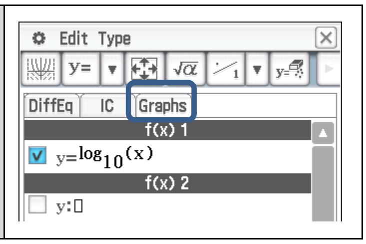


a) What are the key features of this curve?

You might guess that it looks like a log function.

Test your function

- Tap the Graphs tab
- Enter the function log(10,x)
- Tap



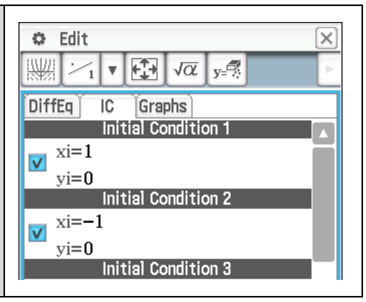
If your function is correct it will be drawn in the same position.

b) Use guess and check to determine the function with derivative of $\frac{1}{x}$

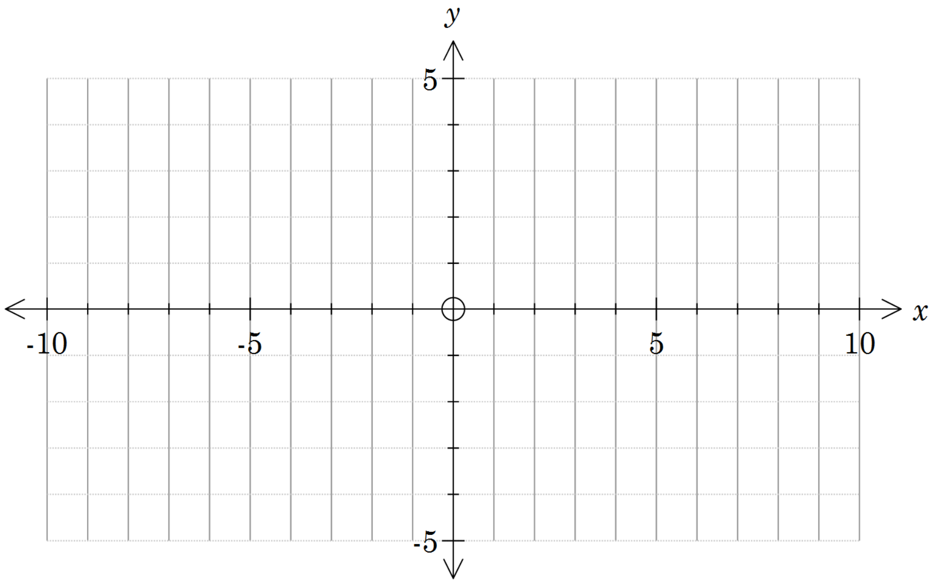
c) You can see that the slope field is symmetrical about the *y*-axis. Add a second initial condition (–1, 0)

Enter initial condition

- Tap on IC tab
- Set a second initial point to (–1, 0)
- Tap



Record the resultant curve(s).



d) From your work predict $\int \frac{1}{x} dx$

Learning notes

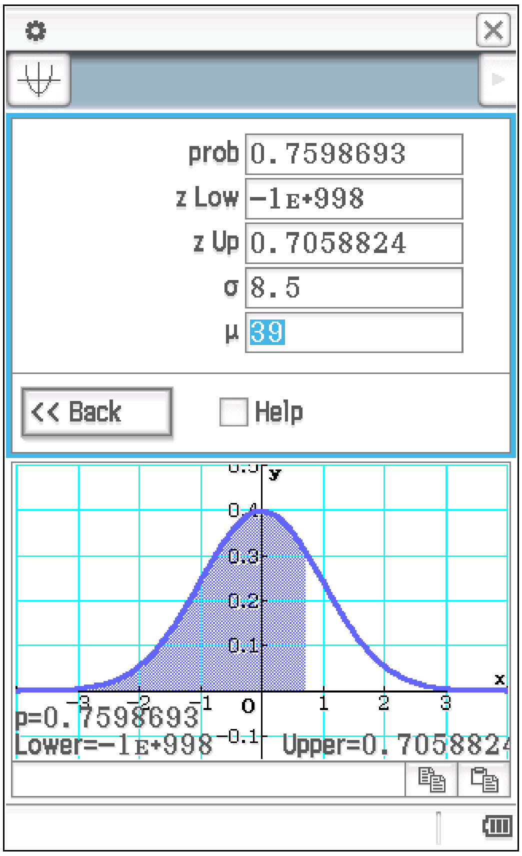
In the previous activity you found $\frac{d}{dx}\ln(x) = \frac{1}{x}, \quad x > 0$

Here you are working from the gradient and looking at what the function leading to that gradient might be. This is a differential equation (an equation involving a derivative). However $\frac{1}{x}$ is defined for $x \neq 0$. The investigation in this activity should help explain why $\int\left(\frac{1}{x}\right)dx = \ln(x) + \text{c}, \ x > 0$ and $\int\left(\frac{1}{x}\right)dx = \ln(-x) + \text{c}, \ x < 0$

This result is often summarised as $\int\left(\frac{1}{x}\right)dx = \ln|x| + \text{c}$

Chapter 5 Continuous random variables

Activity	ClassPad applications	Key concepts
Uniform distribution	Statistics Main	Use relative frequencies to estimate probabilities associated with continuous random variables.
Calculating with Continuous Random Variables	Main	Understand probability density functions and probabilities associated with continuous random variables
Non-uniform continuous random variables	Statistics Main	Model grouped data with probability density functions
Normal CD	Statistics Main E-Activity	Calculate probabilities for normal distributions.
Continuous Distributions	Statistics Main	Use area to calculate statistics for continuous distributions.



Activity 30 Uniform distribution

Aim: Use relative frequencies to estimate probabilities associated with continuous random variables.

Quang's class were asked to find out how far they could walk/run in 30 seconds. Some walked, some ran. The distances were then rounded to the nearest metre and summarised below.

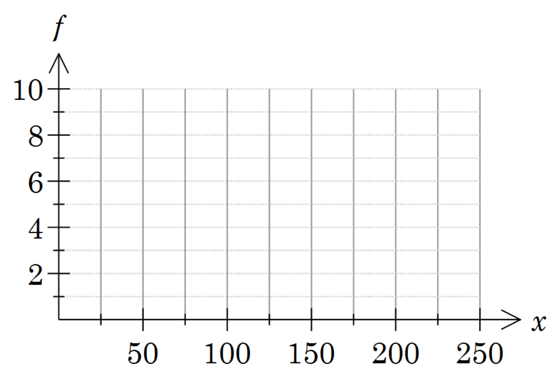
Interval	Frequency
0 – 49	0
50 – 99	6
100 –149	6
150 – 199	6
200 – 250	6

Interval	Cumulative frequency
≤ 49	
≤ 99	
≤ 149	
≤ 199	
≤ 249	

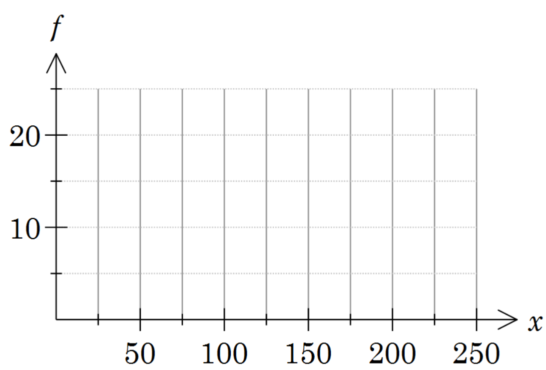
1. Complete the cumulative frequency table above.

2. Graph the data.

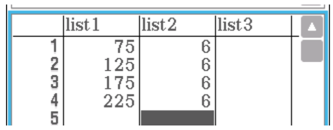
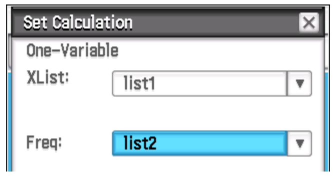
Frequency histogram



Cumulative frequency



3. Determine the

Calculate One-variable statistics • Open Statistics app • Enter the interval midpoints 75, 125, … in list1 and frequency in list2	
• [Calc | One-Variable] • Set XList and Freq as shown and tap OK	

a) mean

b) standard deviation

4. Based on this data estimate the probability that a student selected at random from Quang's class covered

a) between 50 and 100 metres

b) between 50 and 60 metres

c) between 76 and 77 metres

d) a recorded distance of 123 metres (rounded to the nearest metre)

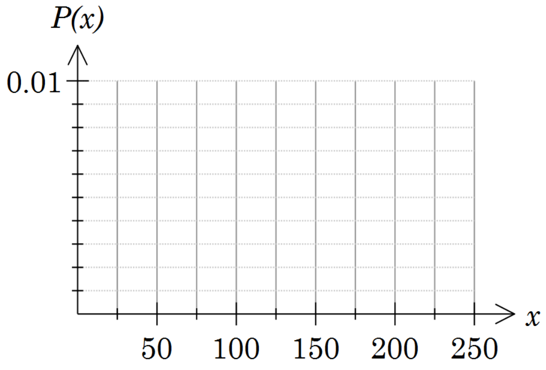
e) between 80.1 and 80.2 metres

f) exactly 100 metres

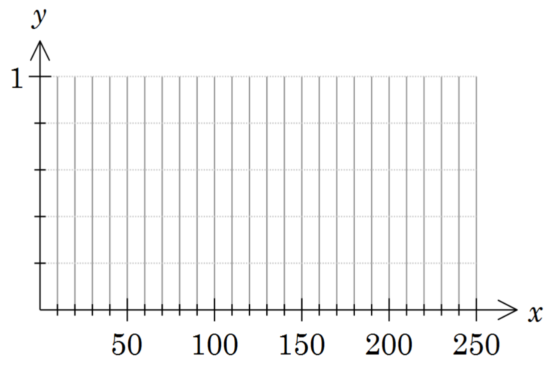
g) less than 205 metres.

5. a) Draw graphs for both the probability density function (PD) and the cumulative density function (CD)

(PD)



(CD)



a) State the probability density function explicitly, including the domain. (I.e. the equation of your graph)

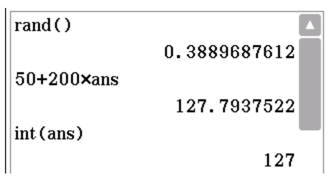
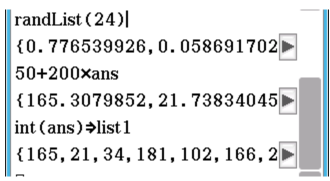
$P(X = x) =$

b) State the cumulative distribution function explicitly

$C(X < x) =$

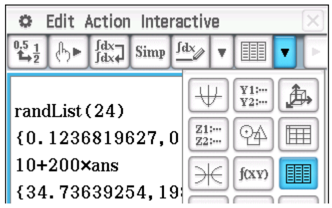
c) Calculate $C(X < 205)$ and interpret the answer.

Create a simulation for the distances travelled.

Setup • Open Main √α and set to Decimal mode	
Generate a random number between 50 and 200 • Open Keyboard select abc • Enter rand() and press EXE to generate a random number between 0 and 1 • Multiply previous answer by 200 and add 50 • Truncate the number to an integer.	
Generate a random list of 24 numbers between 50 and 200 • Edit first line to randList(24) and press EXE • Edit the last line to store in list1	

Open Statistics window

- Select Statistics from the applications pull-down menu
  A half screen Statistics window will appear with the 24 random numbers in list1

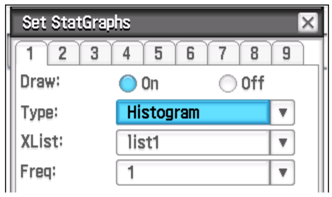


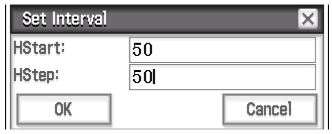
Draw graph

- Set the StatGraph1 to a Histogram as shown
- Tap Set

- Tap to draw the graph
- Set Hstart: and HStep as shown
- Tap OK

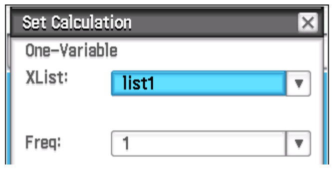
- [Analysis | Trace] to obtain frequencies for grouped data





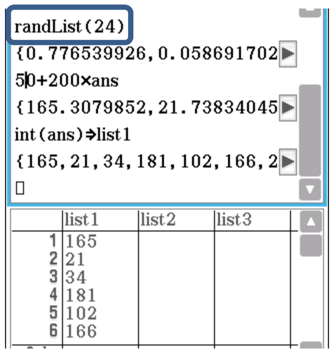
Calculate mean and standard deviation

- [Calc | One-Variable]



To rerun simulation

- Tap in Main window
- Tap in the line randList(24) and press EXE
- Tap in Statistics window to see new data
- Redo analysis of results



6. Run your simulation a number of times. For each trial calculate mean and standard deviation (σ). To obtain the frequencies for each trial: draw a histogram with Hstart: 50 and HStep: 50 and use [Analysis | Trace].

			Frequency			
Trial	mean	σ.	50 – 99	100 –149	150 – 199	200 – 249
1						
2						
3						
4						
5						

Learning notes

This situation refers to data that is continuous, i.e. the measurement recorded is always rounded to some degree.

For discrete random variables we know that the sum of the probabilities for all possible events is 1. This is equivalent to the area of the probability histogram equalling 1. For continuous random variables the area under the probability density function curve is 1.

For this scenario it is a reasonable assumption that the distances travelled by the students was evenly spread out over a distance of 200 metres and each distance is equally likely, i.e the probability of any 1 metre interval is $\frac{1}{200}$.

Q3 To calculate mean and standard deviation assume all results in that interval are the midpoint of the interval.

The answer to Q4 f) is zero. For a continuous distribution, probabilities are calculated for a range of values and equal the area beneath the probability density function.

For the simulation, the command RandList(24,50,249) will generate the list of numbers. However the slower process outlined in the activity is used to emphasise the concept of a continuous distribution and that rounding is inherent in the measurement.

Activity 31 Calculating with continuous random variables

Aim: Understand probability density function and probabilities associated with continuous random variables.

In the previous activity a uniform distribution was explored. What if the distribution is not uniform?

1. Consider a probability density function that is quadratic with a minimum value of 0 and maximum of 2, i.e. $P(X=x)=kx\left(2-x\right), 0\le x\le 2$.

   a) Solve $\int_0^2 kx(2-x)dx=1$ for k (the sum of the probabilities must be 1).

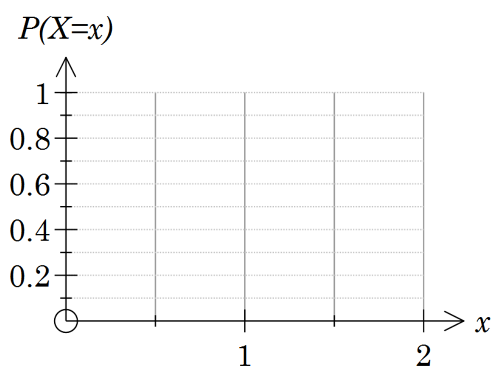
   b) Calculate the expected value.
   (see Learning notes for formulae)

   c) Calculate the standard deviation.

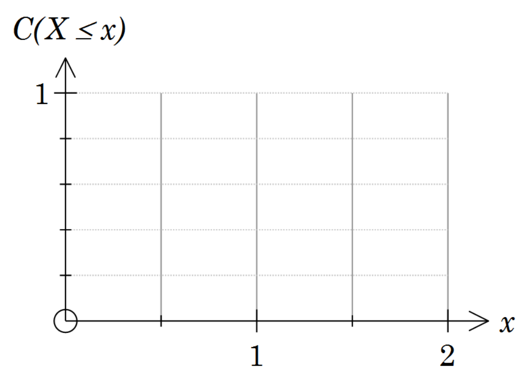
   d) Determine the cumulative distribution function $C(X\le x)=\int_0^x f(t)dt$

   e) Draw the graphs.

   Probability density function

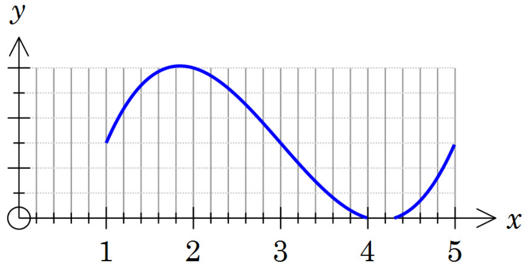


   Cumulative distribution function



   f) What is the probability of $0.5<X<1.5$?

2. Consider a probability density function that is a cubic of the form
$P(X=x)=k\left[(x-1)(x-3)(x-5)+3\right],\ 1\le x\le 5$



a) What is the minimum possible score?

b) Determine k

c) Determine the probability of a score:

(i) between 1 and 2

(ii) less than 3

(iii) more than 2

Learning notes

Q1a) When using the integral template, tap right after the dx to exit the template and enter =1.

Random variables: calculations summary

	Discrete random variables		Continuous random variables
	list	Grouped	
The sum of probabilities is 1	$\sum p_i = 1$	$\sum p_i f_i = 1$	$\int f(x)dx = 1$
Expected value $E(X)$	$E(X)=\sum x$	$E(X)=\sum_x xf(x)$	$E(X)=\int xf(x)dx$
Variance (σ^2)	$\sum_x (x-E(X))^2$	$\sum_x (x-E(X))^2 f(x)$	$\int_l^r (x-E(X))^2 f(x)dx$
Standard deviation (σ)	$\sqrt{\sum_x (x-E(X))^2}$	$\sqrt{\sum_x (x-E(X))^2 f(x)}$	$\sqrt{\int_l^r (x-E(X))^2 f(x)dx}$

Activity 32 Non-uniform continuous random variables

Aim: Model grouped data with probability density functions.

1. Raquel's class repeated the experiment in the activity *Uniform distribution* with different results, shown below.

Interval	Frequency
50 - 99	2
100 - 149	10
150 - 199	10
200 - 250	2

Interval	Cumulative frequency
≤ 99	
≤ 149	
≤ 199	
≤ 249	

In Statistics

a) Determine the

(i) mean

(ii) standard deviation

b) Complete the cumulative frequency table above.

c) Raquel looks at this data and suggests a parabola could be used to model the distribution.

(i) Use a quadratic regression to determine a suitable model for the frequency.

(ii) Locate the roots of this model.

(iii) Calculate the area between the model and the x-axis.

(iv) Determine the quadratic probability density function for this distribution.

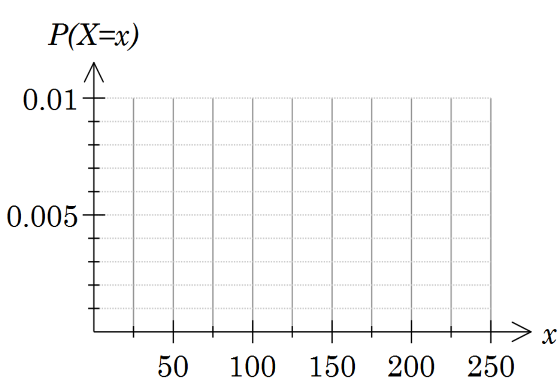
(v) Determine the Cumulative distribution function for your model.

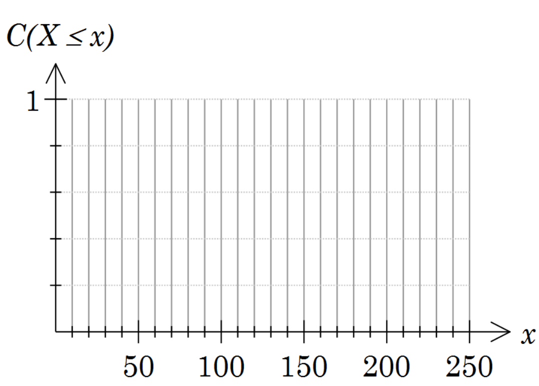
d) Based on your model, estimate the probability of a student selected at random covering

(i) between 50 and 99 metres.

(ii) between 110 and 125 metres.

2. a) Draw graphs of the probability density and the cumulative distribution functions.





b) Determine the

(i) expected value

(ii) variance

(iii) standard deviation.

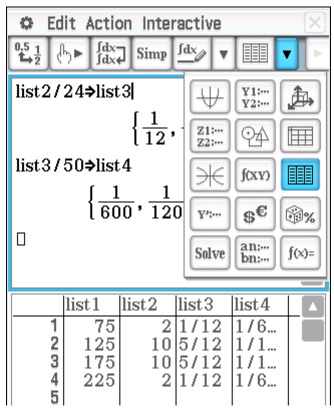
3. Discuss the limitations of the quadratic model you have worked with.

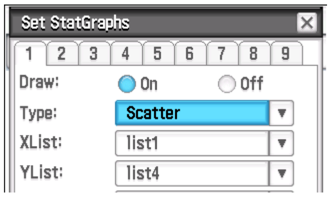
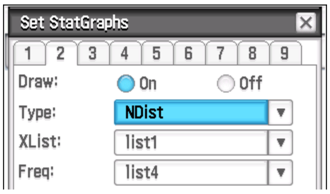
4. Create a normal distribution model for the data.

Probability

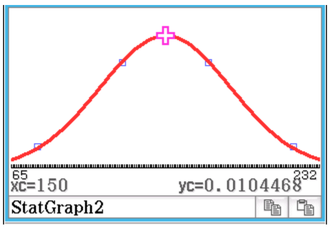
Convert frequency to a probability

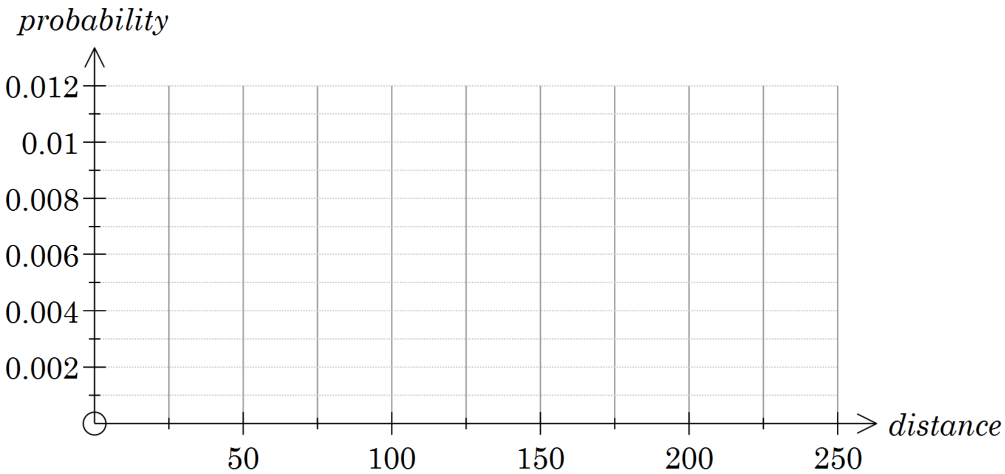
- Open Main $\sqrt{\alpha}$
- Enter the statement list2 / 24 and store to list3
  This is the relative frequency for each interval
- Enter list3 / 50, i.e. $\dfrac{\text{frequency}}{\text{width of interval}}$ and store in list 4



• Open Statistics from the apps pull-down menu The new lists should now be present	
Draw graph • Tap [icon] • Set StatGraph1 to a Scatterplot of list1 v list4	
• Set StatGraph2 to a Normal distribution curve of list 1 v list4 • Ensure Draw is set to On • Tap [icon]	

a) Record the graph

Read values from graph • In the graph window • [Analysis | Trace] • Use up/down button to select the curve • Use left/right to move along graph	



b) Describe the shape of the curve.

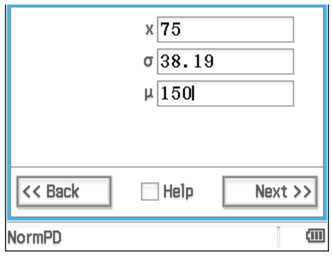
c) Complete the table of values by reading the graph

Distance (x)	75	90	105	120	150	180	195	210
P(X=x)								

5. Use ClassPad's normPD

Calculate PD values

- Tap in the Statistics window
- [Calc | Distribution]
- Select Normal PD
- Tap Next >>
- Tick the Help button and enter the appropriate values (Use your answers from Q1 a) for mean and standard deviation)
- Tap Next >>

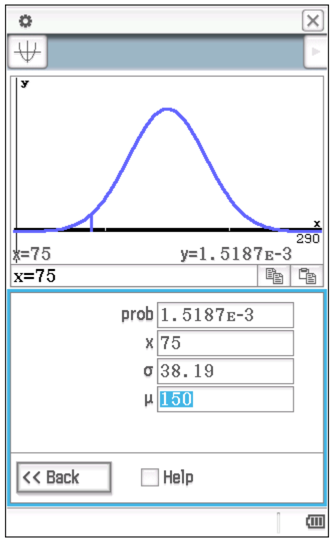


Draw Graph

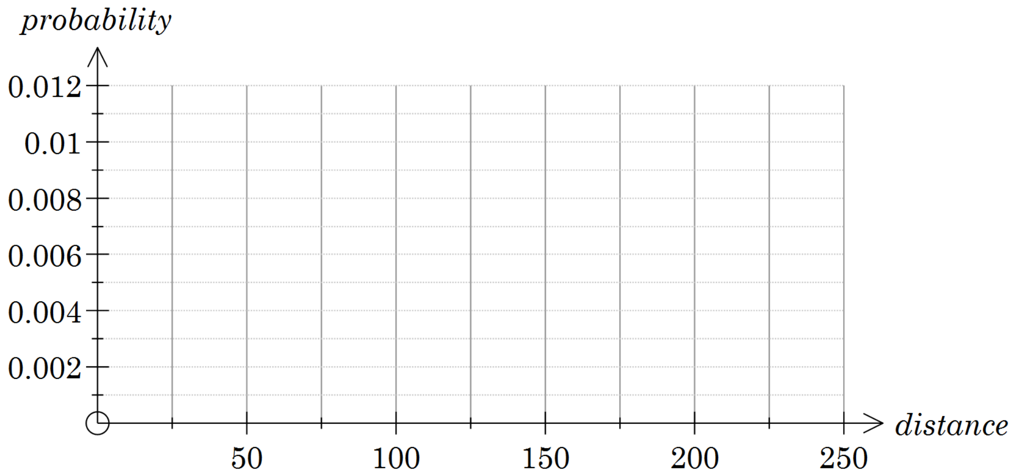
- Tap [graph icon] to draw the graph

Change *x*-value

- [Analysis | Trace] to estimate probability function values for other *x*-values
  OR
  Tap << Back and change the *x*-value and tap Next >>



a) Record the graph.



b) Complete the table of values by reading the graph.

distance	75	90	105	120	150	180	195	210
Probability								

c) How do your answers compare to Q4?

Learning notes

Q1 a) Use Statistics app to answer questions. Refer to the activity Uniform distribution for instructions if needed.

Q1 c) Hints to make calculations easier:

- [Calc | Regression | Quadratic Reg] and store the regression equation in y1 so you can then work with the equation easily in Main
- Store the roots as variables, so you can easily refer to these later
- Define the PDf function. You may also want to restrict the domain using your stored values for the roots

- Check that the area under the curve is 1
- Switch variable to define the CD function e.g. $\int_{lower}^{x} P(y)dy$
- Part d) can done by integration of the PD function or using the difference in Cumulative Distribution values at the end points
  e.g. $\int_{110}^{125} P(x)dx$ or $C(125)-C(110)$ are equivalent expressions

Q2 a) Sketch the graphs by opening a graph window from Main and dragging the functions into the window, then adjust the window as required.

Q2 b) Refer to previous Activity's Learning notes for the formulae to calculate expected value and standard deviation.

Q4 a) Make sure you use the up and down arrows to choose the normal distribution curve. It is likely the default curve will be the regression, then the scatter graph and finally the normal distribution curve.

Activity 33 Normal CD

Aim: Calculate probabilities for normal distributions.

Normal distributions are usually described in terms of the mean and variance: $N(\bar{x},\sigma^2)$. ClassPad uses the mean and standard deviation.

1. Use the mean and standard deviation from Q1 in the previous activity *Non-uniform continuous random variables* to calculate the probability that a student randomly selected from Raquel's class covered:
(assume a normal distribution and see Learning notes for instructions)

   a) between 50 and 99 metres

   b) between 110 and 125 metres

2. Mia believes the length of 220 mm pavers is distributed normally with a mean of 221 mm and standard deviation of 1.1 mm.

   a) Determine the probability that the length of a randomly selected paver is:

      (i) less than 220 mm

      (ii) more than 223 mm

      (iii) between 219 and 223 mm

      (iv) exactly 221 mm

      (v) more than 223 mm given that is longer than 221 mm.

   b) Determine the 90th percentile, i.e. 90% of the pavers should be less than this length.

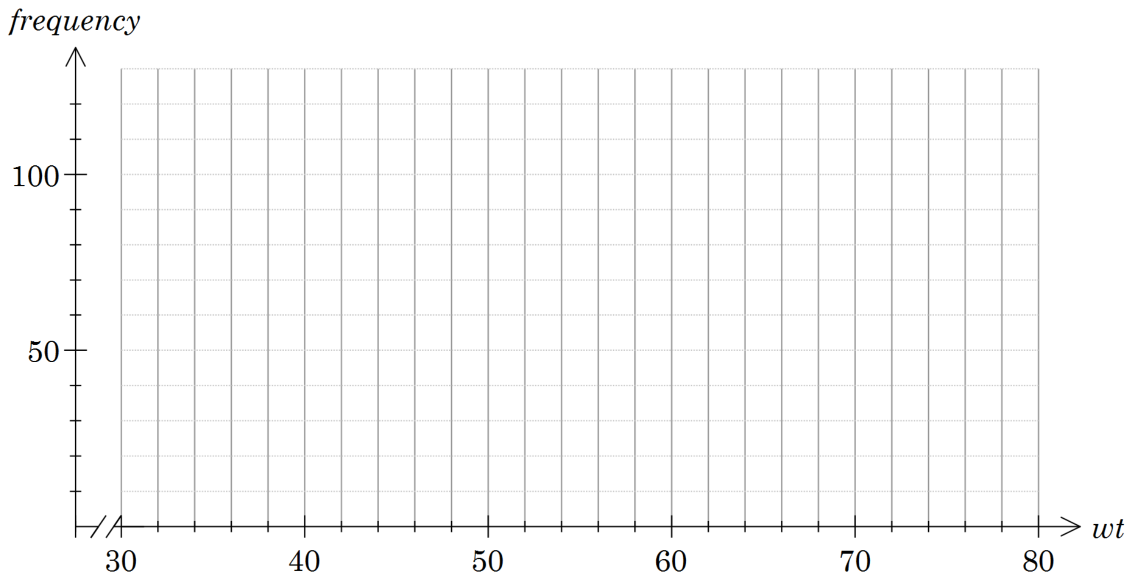
   c) Determine the interquartile range (the difference between the third and first quartiles).

3. Matt collects data on a sample of the eggs produced at his farm. His results are shown in the table.

Class interval	Class centre (g)	Frequency
<42	38	6
42.1–50	46	74
50.1–58	54	102
58.1–66	62	122
66.1–74	70	85
>74	78	11

a) Calculate the mean and standard deviation of the sample.
(Assume that all eggs in an interval have an average weight equal to the class centre.)

b) Plot a histogram and explain why a normal distribution is an appropriate model for this data.



c) Use the normal distribution model to determine the probability that a randomly selected egg weighs:

(i) more than 45 g

(ii) less than 53 g

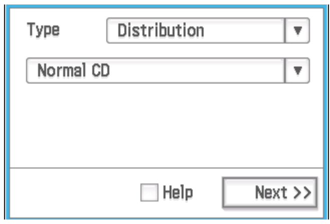
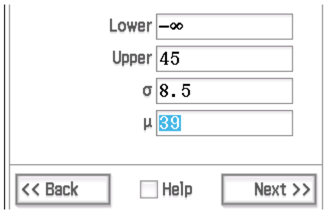
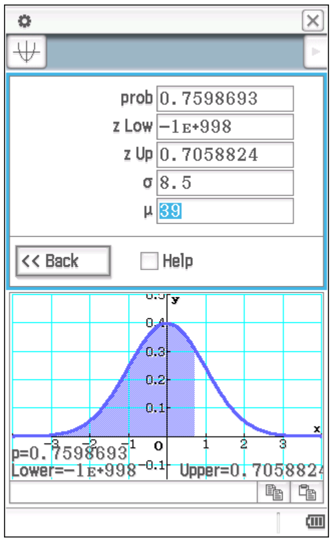
(iii) between 69 g and 70 g

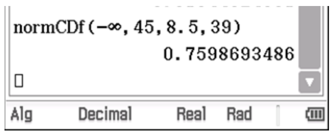
(iv) less than 66 g given it weighs more than 45 g.

Learning notes

There are several ways you may calculate probabilities involving the normal distribution on ClassPad.

In statistics.
The advantage in using statistics is being able to draw a graph and thus confirm the result is what you expect. To calculate $P(X < 45)$, for $N(39, 8.5^2)$

Statistics • Open Statistics app • Select [Calc | Distribution] • Select Normal CD • Tap Next >>	
• Set the parameters for the problem Lower boundary (−∞ if there is no bound) Upper boundary Standard deviation and Mean • Tap Next >>	
• Tap the graph icon to graph the distribution and region for which the probability is being calculated. • Ticking the Help checkbox can be helpful	

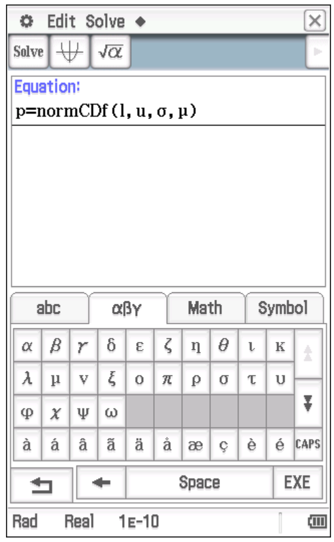
In Main • Open Main $\sqrt{\alpha}$ • [Interactive | Distribution/Inv. Dist | Continuous | normCDf] • Enter appropriate values.	

Use Main when you want to combine results. For example calculating a probability that involves both ends of the distribution.

Create and use an e-Activity

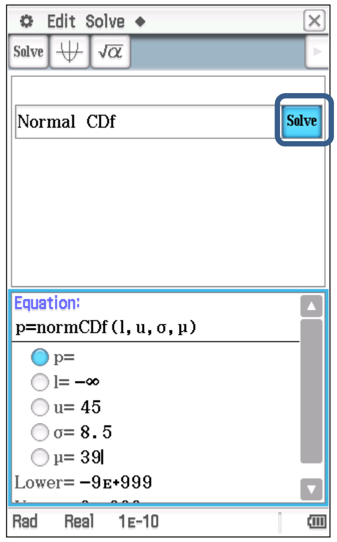
Create eActivity

- Open the eActivity app
- Select [File | New]
- Select [Insert | Strip(2) |NumSolve]
- Enter the equation: as shown
  Use the soft Keyboard for entering the symbols.
  The symbols used inside the bracket are a choice.
  normCD is case sensitive.
- Press EXE



Solve

- Tap Solve
- Put values in for any 4 of the variables and check the radio button next to the variable you want to calculate.
- Put a title in the NumSolve strip that is meaningful to you
- Tap [save icon] to save.
- Note: you can add other strips to your eActivity.



An advantage of the numSolve strip in the eActivity is that it incorporates the inverse function as well.
In Statistics we would require invNormCDf to find the limit for a given probability.

Activity 34 Continuous distributions

Aim: Use area to calculate statistics for continuous distributions.

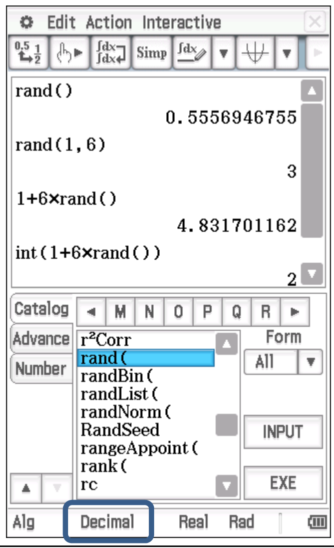
Consider a uniform distribution: e.g. roll a fair six-sided die. We expect the probability of each individual outcome to be the same and to equal $\frac{1}{6}$.

For a continuous random variable we refer to a density function. Probabilities are calculated over an interval.

1. Complete the table.
   Generate sufficient examples to confidently describe the output for each ClassPad statement.

Generate random numbers

- In Main
- Ensure ClassPad is in Decimal mode
- Enter the commands as shown below
  - Type rand from the alphabet keyboard
    OR
  - Use the catalogue
- Press EXE
- Run each command a number of times



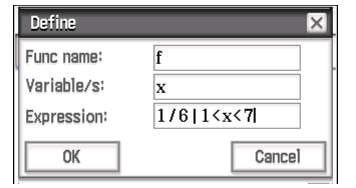
ClassPad Statement	Examples of ClassPad Output	Description of command
rand		Generates a 10 digit decimal between 0 and 1
rand(1,6)		
1+6×rand()		
int(1+6×rand())		
int(7 – 6×rand())		

2. Consider the ClassPad command 1+6×rand().

The output X is close to a continuous random variable.

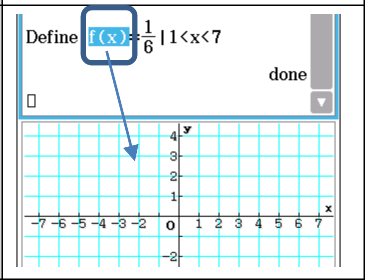
Define the distribution

- Select [Interactive | Define] and enter the function, as shown

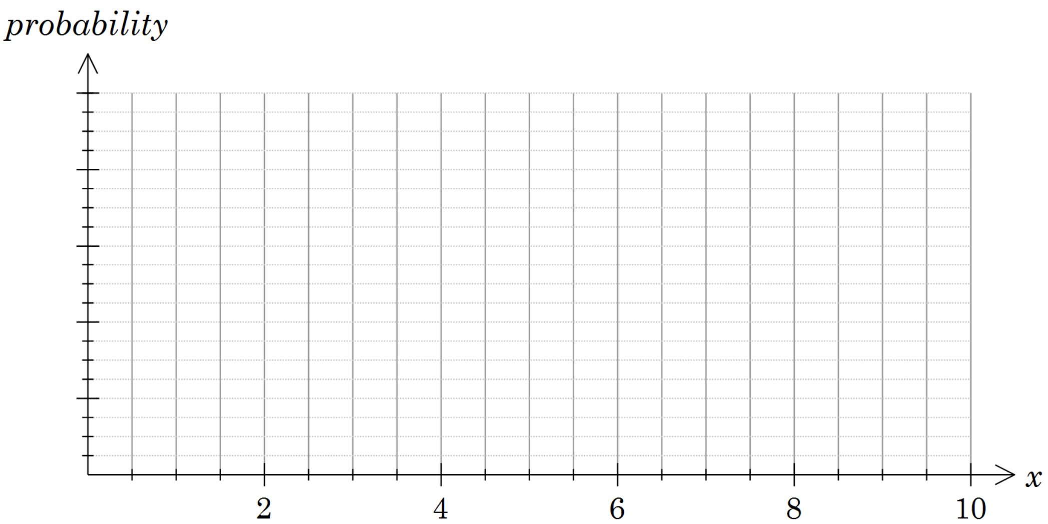


Graph the function

- Tap to open the graph window (select from the app pull down menu)
- Highlight f(x)
- Drag into the graph window
- Tap to adjust the view window



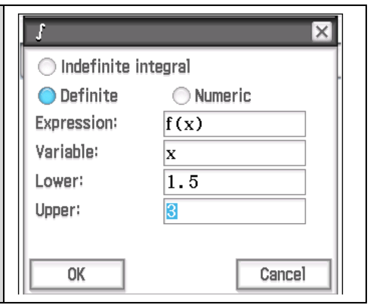
a) Draw a graph of the distribution.



b) What is the area between the x-axis and the above graph between $x = 1.5$ and $x = 3$, i.e. $1.5 < X < 3$.

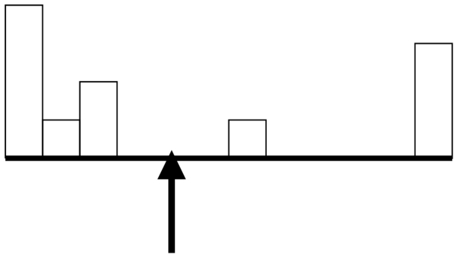
Calculate integral

- [Interactive | Calculation | ∫]
- Select Definite
- Enter the appropriate values
- Tap OK

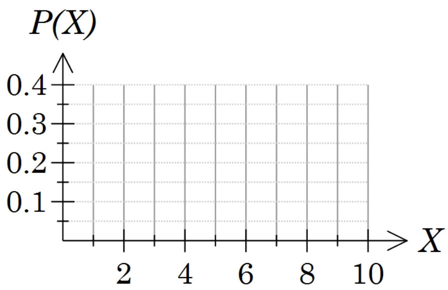
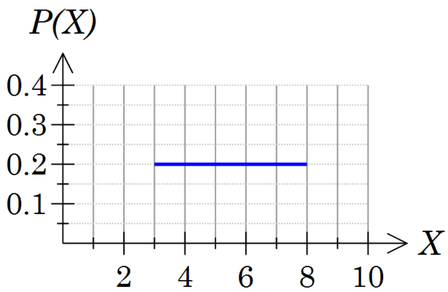
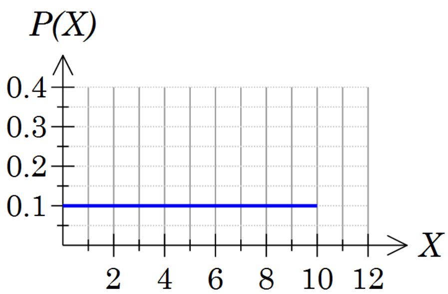


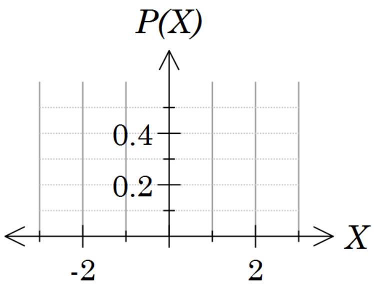
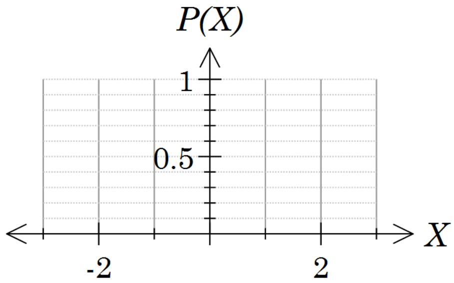
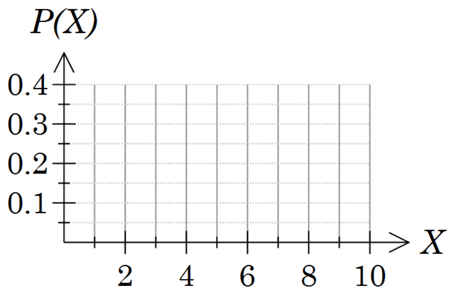
c) Calculate the probability of the command generating a value between 1.5 and 3.

d) Calculate the mean of the distribution $f(x)$. (The mean is the balance point for the distribution and is the expected value.)



3. Complete the table. Refer to the learning notes for formulae and construction details.

	Probability density function	Graph	Mean	Standard deviation
a)	$P(X)=\begin{cases}0.2 & 0\le X\le 5\\ 0 & \text{elsewhere}\end{cases}$			
b)				
c)				

d)	$P(X)=\begin{cases}\frac{12-3X^2}{32} & -2\le X\le 2\\ 0 & \text{elsewhere}\end{cases}$ 		
e)	$P(X)=\begin{cases}\frac{3X^2}{16} & -2\le X\le 2\\ 0 & \text{elsewhere}\end{cases}$ 		
f)	$P(X)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(X-5)^2}$ See Learning notes. 		

4. Referring to the table in Q3, write a brief statement to account for each of the following. Use the graphs and refer to the distribution as shown in the graphs.

   a) Parts a) and b) have the same spread (standard deviation) but different means.

   b) Parts c) has double the standard deviation of parts a) and b).

   c) Part e) has a greater standard deviation than d).

5. EXTENSION

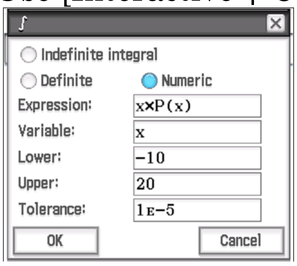
Show that the standard deviation for the uniform distribution of a continuous random variable X on the interval (a,b) $p(X)=\frac{1}{b-a}, a\le X\le b$ is $\sigma=\frac{b-a}{\sqrt{12}}$

Learning notes

Random number generators, like that programmed into ClassPad, produce an output approximating a continuous distribution. Typically a number between 0 and 1 is generated. The probability of an individual outcome is very small and in a continuous distribution is 0. To generate a dice roll the calculator multiplies the random number by 6, adds 1 and truncates the result (ignores the decimal part of the number).

Formulae for calculating mean and standard deviation.			
	Discrete	Grouped discrete data	Continuous distribution
Mean	$\mu = \frac{\sum x_i}{n}$	$\mu = \frac{\sum x_i f_i}{n}$	$\mu = \int_l^r x \times p(x)\ dx$
Standard deviation	$\sigma = \sqrt{\frac{\sum (x-\mu)^2}{n}}$ $= \sqrt{\frac{\left(\sum x^2\right) - \mu^2}{n}}$	$\sigma = \sqrt{\frac{\sum f_i (x_i - \mu)^2}{n}}$ $= \sqrt{\frac{\sum f_i (x_i)^2 - \mu^2}{n}}$	$\sigma = \sqrt{\int_a^b x^2 p(x)\ dx - \mu^2}$

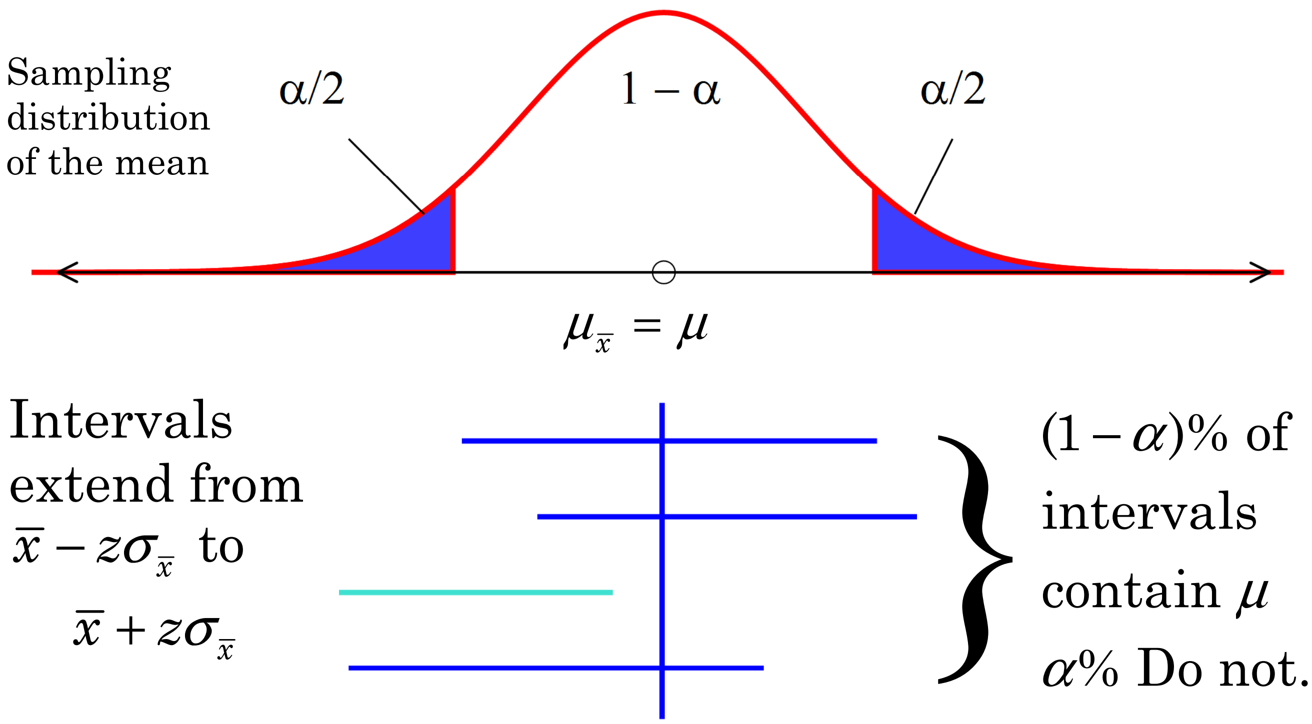
Q3f) This is the formula for the normal distribution. It cannot be integrated algebraically and hence has to be integrated numerically.
Use [Interactive | Calculation | ∫]



Chapter 6 Interval estimates

Activity	ClassPad applications	Key concepts
Simulating random samples	Main Statistics	Use graphical displays of simulated data to investigate the variability of random samples
Sample proportions	Program Main	Simulate repeated random sampling and explore the distribution of sample proportions
Confidence intervals for proportions	Main Statistics	Calculate confidence intervals

Intervals and levels of confidence



Activity 35 Simulating random samples

Aim: Use graphical displays of simulated data to investigate the variability of random samples.

Generate lists of simulated data from different distributions.

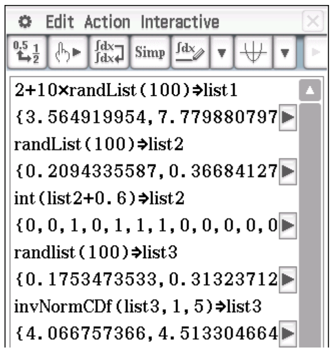
1. Explore the output from the following ClassPad commands to complete the table. For distribution: write Uniform, Normal, Bernoulli, Binomial, Other.

ClassPad expression	Distribution	Range (may be approximate)
2+10×rand()		
intg(rand()+0.6)		
invNormCDf(rand(),1,5)		

2. Use the commands in Question 1 to create a sample from each distribution.

Generate samples of 100 and store in Statistics

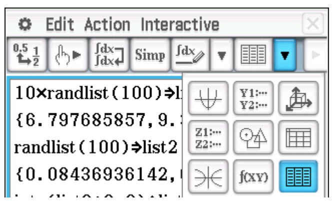
- Open Main
- Enter the command
  2+10×randlist(100) ⇒ list1
- Enter the commands
  randlist(100) ⇒ list2
  int(list2+0.6) ⇒ list2
- Enter the commands
  randlist(100) ⇒ list3
  invNormCDf(list3,1,5) ⇒ list3



a) Draw histograms of the distributions to show the shape of the distributions.

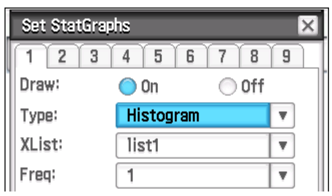
Display Histograms

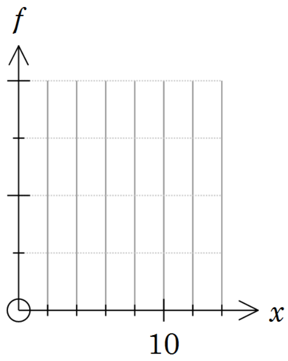
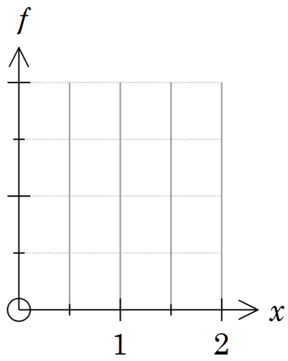
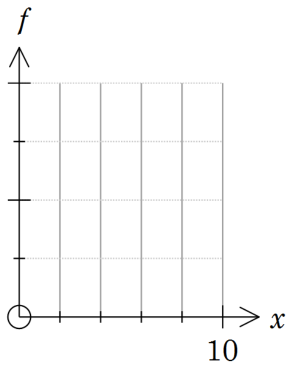
- Tap [icon] from the apps pull-down menu to open a Statistics window



Set up and draw graph

- Tap [icon] and set parameters
- Tap [icon] to draw the graph
- Adjust XList: to draw list2 or list3

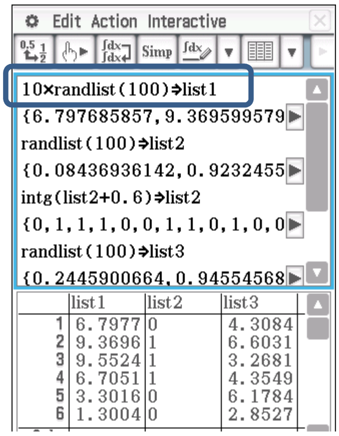


List 1	List 2	List 3
		

b)

Generate new lists and redraw the graphs

- Close the graph window
- Tap in Main
- Scroll up to the line 10×randlist(100) ⇒ list1 and press EXE to regenerate the lists
- Tap in the Statistics window and redraw the graphs as done previously

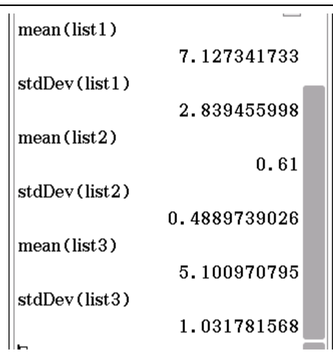


Describe what is similar and what changes.

c) Calculate means and standard deviations for each distribution.

Calculate mean and standard deviation

- Enter the commands to calculate mean and standard deviation for each list
  (Use the catalogue or soft keyboard)
  Omit standard deviation if your ClassPad is taking too long to do the calculations.
- Scroll up to the line 10×randlist(100) ⇒ list1 and press EXE to regenerate the lists



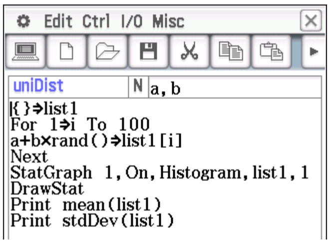
Populate the table by running the simulations a number of times.

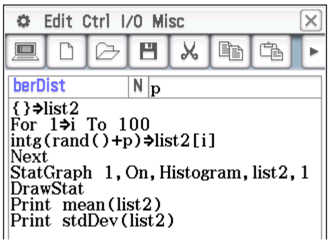
	List 1		List 2		List 3	
Trial	Mean	S.D	Mean	S.D	Mean	S.D
1						
2						
3						
4						
5						

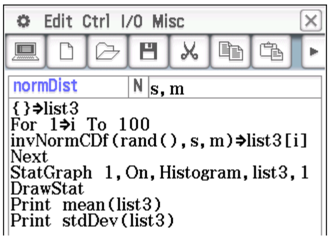
d) How do your results in c) compare to what you would expect?

EXTENSION:

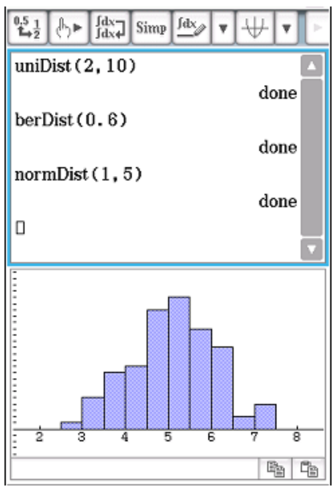
You can write programs to automate the processes above. Each screenshot shows a program for one of the distributions, displaying the histogram, the mean and standard deviation.







You may wish to omit standard deviation to speed up the program execution.

You can run the programs from within Main, as shown in the screenshot You can also adjust the parameters and explore how those changes affect the sample.	

Learning notes

This activity makes use of the ClassPad's randlist function to create three different distributions and being able to recalculate all the commands in Main to regenerate those lists, i.e. get a new sample of that distribution.

The commands:

2+10×randlist(100) generates random numbers between 2 and 12. There is a list of 100 of these numbers constituting a random sample of 100 from the uniform distribution. The theoretical mean is 7 and from earlier work the standard deviation is $\sqrt{\frac{1}{12}(12-2)^2} \approx 2.89$.

int(list2+0.6) ⇒ list2 adds 0.6 to the random number between 0 and 1 and then truncates the decimal part leaving either 0 or 1. This is a Bernoulli event with a 1 expected 60% of the time, i.e. a mean of 0.6.

invNormCDf(list3,1,5) ⇒ list3 generates a number based on the position in the normal distribution curve with standard deviation 1 and mean 5. A value of 0.5 generates the mean value, a value of 0.9 calculates the 90th percentile.

The emphasis in this activity is appreciating that the sample mean and sample standard deviation vary between samples.

Activity 36 Sample proportions

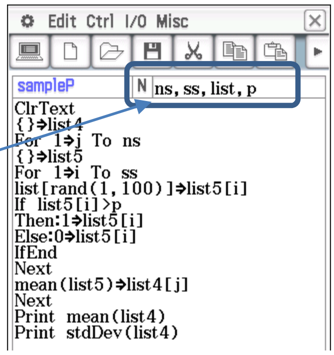
Aim: Simulate repeated random sampling and explore the distribution of sample proportions.

A sample proportion can be thought of as the percentage of the sample that meets a set of conditions.

In order to investigate sample proportions you will first write a program to collect a sample from the lists generated in the previous Activity. That is, we will be sampling from an approximately uniform distribution (list1), a Bernoulli distribution (list2) and an approximately normal distribution (list3).

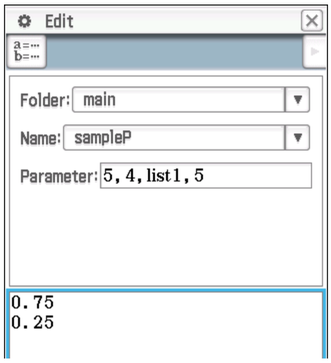
Enter Program

- Open Program app
- Tap [🗋] to start a new program
- Enter the name *sampleP*
- Enter the text as shown in the adjacent screenshot
  Don't forget the parameters
- Tap [💾] to save the program
- Tap [💻] to exit the editor



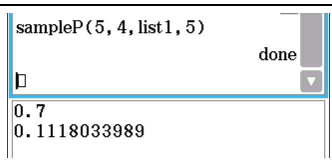
Run the program

- Tap [💻] to return to the opening screen
- Enter the parameters required:
  number of samples, sample size, the list with the source of the data, and the target proportion.
- Tap [▶] to run the program
  In the screen shot 5 samples of 4 results from list1 are collected. The mean and standard deviation for the proportion of these scores greater than 5 is displayed in the Program output window.



Run the program from Main

- Open $\sqrt{\alpha}$ (Main)
- Enter the program name *sampleP* and enter the 4 parameters.

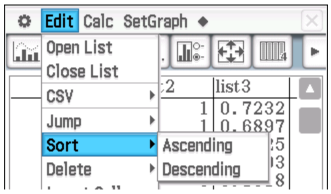


The program provides a tool for collecting sample proportions. The activity requires you to investigate the distribution of the sample proportions.

1. Record the population proportion of scores greater than 5 for list1 and list3.

Determine population proportion

- Open Statistics app
- [Edit | Sort | Descending]
- How many lists: select 1 and tap OK
- Select List Name: e.g. list1 and tap OK
- Scroll down list to determine how many of the 100 results are greater than 5



2. Investigate the effect of number of samples on the mean and standard deviation of $\hat{p}$.

Collect some data (using different numbers of samples) and populate the tables below.

Data source: list1
Sample size: 4
Proportion greater than: 5

No. of samples	Mean of $\hat{p}$	Standard deviation of $\hat{p}$

Data source: list3
Sample size: 4
Proportion greater than: 5

No. of samples	Mean of $\hat{p}$	Standard deviation of $\hat{p}$

Describe your findings.

3. Investigate the effect of sample size on the mean and standard deviation of $\hat{p}$.

Collect some data (using varying sample sizes) and populate the tables below.

Data source: list1	Data source: list3
Number of samples: 4	Number of samples: 4
Proportion greater than: 5	Proportion greater than: 5

Sample size	Mean of $\hat{p}$	Standard deviation of $\hat{p}$

Sample size	Mean of $\hat{p}$	Standard deviation of $\hat{p}$

Describe your findings.

4. Investigate the effect of varying p on the mean and standard deviation of $\hat{p}$.

Data source:	list1	Data source:	list3
Sample size:	4	Sample size:	4
Number of samples:	5	Number of samples:	5

p	Mean of $\hat{p}$	Standard deviation of $\hat{p}$

p	Mean of $\hat{p}$	Standard deviation of $\hat{p}$

Describe your findings.

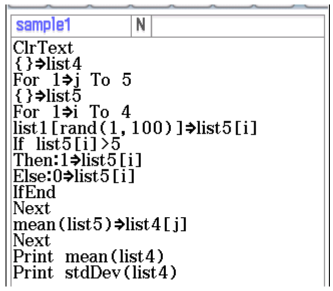
5. Use your results from this activity to comment on the assertion that the distribution of $\hat{p}$ is approximately normal with mean p and standard deviation $\sqrt{\frac{p(1-p)}{n}}$ irrespective of the distribution; n is the sample size.

Learning notes

Many students have problems getting programs to work. This is usually because the code isn't entered correctly. There is a virtual ClassPad file your teacher can download and share with you, enabling you to focus on the problem rather than coding.

It may be easier to understand the program by first writing this program without parameters.

- The program will collect 5 samples.
- Each sample consists of 4 pieces of data from list1, these results are stored in list5.
- The proportion of these greater than 5 is recorded in list4.
- The mean and standard deviation of these proportions is then displayed.



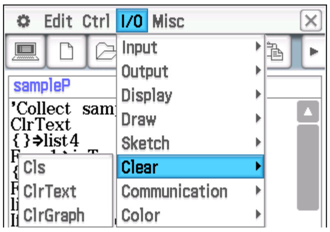
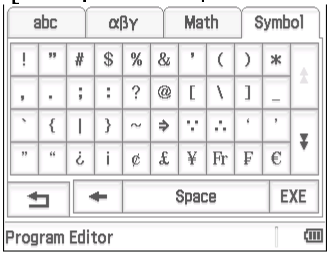
It is more convenient to then edit the program and input the parameters as I the Activity than to edit the code each time a change is required.

The program has 4 parameters:

ns the number of samples to be collected;
ss the sample size or number of results from which to calculate sample proportion;
list the list containing the population we are sampling from; and
p count results greater than this value.

Code	Explanation
ClrText {}⇒list4	Clears the Program output window of any text and empties list4
For 1⇒j To ns	Sets up a loop to collect *ns* samples
{}⇒list5	Empties list 5 ready to store the mean from each sample collected
For 1⇒i To ss	Sets up loop to collect each element of the sample
list[rand(1,100)]⇒list5[i]	Selects a random element from the specified list defined in the parameters
If list5[i]>p Then :1⇒list5[i] Else :0⇒list5[i] IfEnd	Determine if value is greater that the set proportion. If it is, record 1, otherwise record a 0
Next	End of loop for one sample
mean(list5)⇒list4[j]	Store sample mean in list4. List4 will end up with the means for each sample collected.
Next	End of loop go back and collect next sample
Print mean(list4) Print stdDev(list4)	Output results: mean and standard deviation of the sample proportions

Hints:
Use the menus for entering program commands e.g. [I/O | Clear | ClrText]

Use the Symbol tab for { etc.  

Activity 37 Confidence intervals for proportions

Aim: Calculate confidence intervals.

The latest News poll result has Labor's two-party lead at 53-47, up from 52-48 a fortnight ago.

The data for the above headline will have been based upon a survey of a sample from the Australian population. Inference is made from the poll that Labour would win an election if it was held at this time, i.e. an inference is made about the population. How valid or reliable is such a claim?

1. Assume the population is actually 51% two party preferred to Labor. How often would we expect a random sample of 400 people to show:

    a) between 50.5 and 51.5% to Labor?

    b) a Labour win, i.e >50% to Labor?

    c) a 52% lead i.e. 51.5 to 52.5% to Labor?

    d) a 53% lead i.e. 52.5 to 53.5% to Labor?

2. Determine the

    a) 90% confidence interval, i.e. the range of values within which we can expect the population proportion to lie 90% of the time.

    $$\hat{p}-1.65\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p}+1.65\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

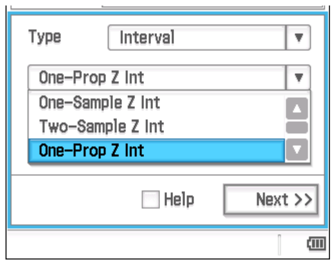
    b) 95% confidence interval, i.e. the range of values within which we can expect the population proportion to lie 95% of the time.

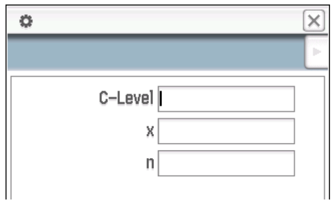
    $$\hat{p}-1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p}+1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

    c) 99% confidence interval, i.e. the range of values within which we can expect the population proportion to lie 99% of the time.

    $$\hat{p}-2.57\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p}+2.57\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

3. Given we don't know the population proportion, what can be inferred about the population proportion from the sample 53-47 quoted at the beginning of the activity? (Assume the original poll surveyed 400 people)

Calculate proportion interval • Open Statistics • [Calc | Interval] • Select One-Prop Z Int • Tick Help • Tap Next	

• Enter values paying attention to help
  o C-Level is confidence level and is entered as a decimal between 0 and 1
  o x is the number of positive results i.e. 0.53×400
  o n is the size of the sample.
• Tap Next
	

a) Determine an approximate 90% confidence interval, i.e. the range of proportions from which our sample proportion of 0.53 might be expected 90% of the time.

b) Determine an approximate 95% confidence interval.

c) Determine an approximate 99% confidence interval.

4. Comment upon this screen shot:

Sample size	Margin of error
400 people	+/- 5%
1100 people	+/- 3%
2500 people	+/- 2%
10,000 people	+/- 1%

Learning notes

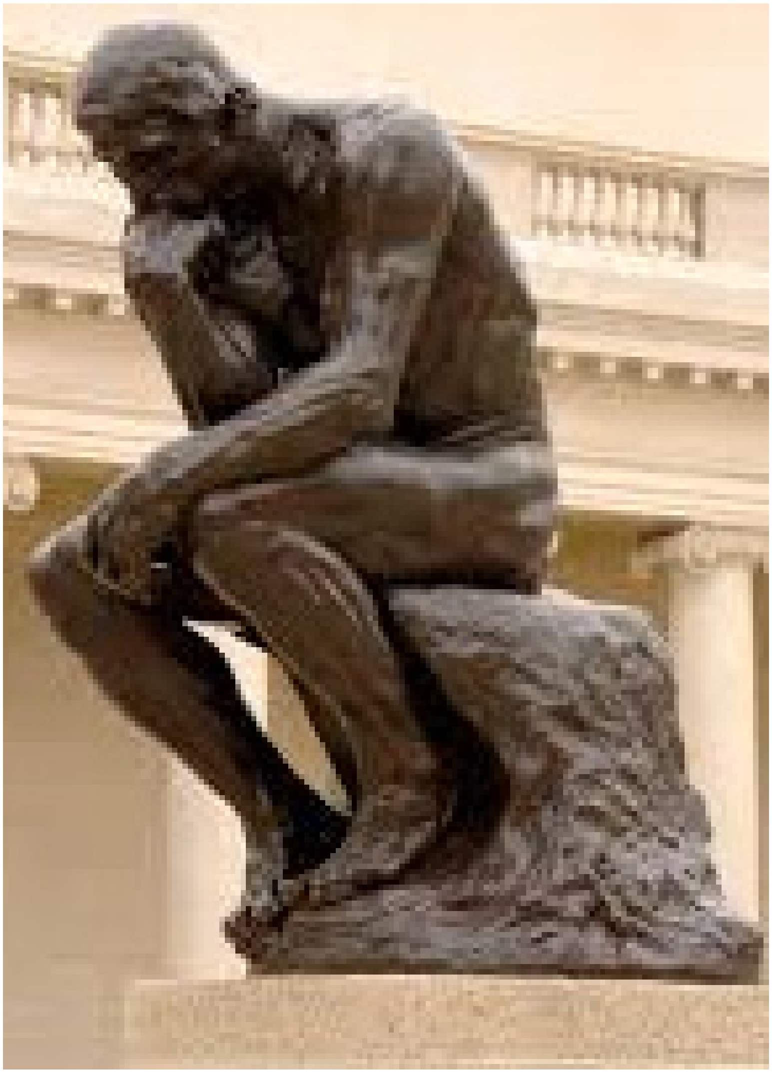
Confidence interval:

Confidence Level	Area between 0 and *z*-score	Area in one tail (α/2)	z-score
50%	0.2500	0.2500	0.674
80%	0.4000	0.1000	1.282
90%	0.4500	0.0500	1.645
95%	0.4750	0.0250	1.960
98%	0.4900	0.0100	2.326
99%	0.4950	0.0050	2.576

Chapter 7 Problems

Looking at limits	Graph&Table Spreadsheet Main	Establish the $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$ numerically and graphically
Sine of x on x	Geometry	Establish the $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$ geometrically
Modelling motion	Main Statistics	Model motion along a straight line

These Activities can be seen as extensions, i.e. providing opportunities for you to challenge yourself with more complex problems. These challenges can consolidate your understanding of the course content.



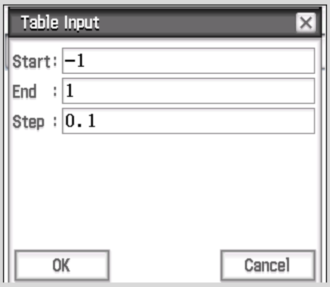
Activity 38 Looking at limits

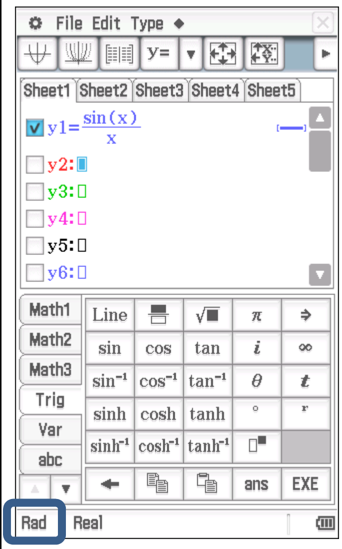
Aim: Establish the $\lim_{x \to 0} \frac{\sin x}{x}$ numerically and graphically.

Using a table of values

Create a table of values for $y = \frac{\sin x}{x}$

- Ensure the angle measurement of your ClassPad is set to radians
- Open Graph&Table app
- Enter the function $y = \frac{\sin x}{x}$
- Tap [icon] to access the Table Input window. Set values as shown and tap OK
- Tap [icon] to display the table and scroll through the values





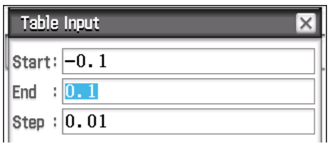
1. What is the value of y when $x = 0$?

2. What is the behaviour of the function as

   a) $x \to 0$ from the negative side?

   b) $x \to 0$ from the positive side?

3. Zoom in by changing the Table Input to:



   a) What can you say about the behaviour of the function as x approaches 0?

   b) Continue to zoom in.

   What do you observe?

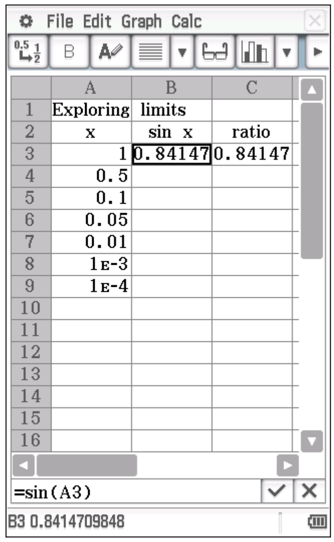
4. State a conjecture for $\lim_{x \to 0} \dfrac{\sin x}{x}$.

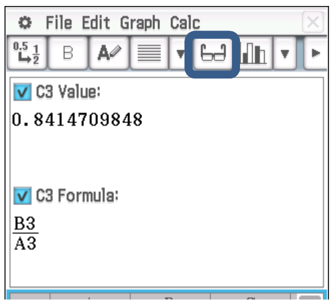
Using Spreadsheet

An alternative approach is the Spreadsheet. Whereas with the table the *x*-values are an arithmetic sequence, any values can be set in the spreadsheet. Setting appropriate values allows you to see the trend as you zoom in.

Build spreadsheet

- Open Spreadsheet app
- [Edit | Clear All] (if required)
- Enter labels in rows 1 and 2, as shown. (optional but useful to identify the purpose of the spreadsheet)
- Enter a list of *x* values in column A like 0.1, 0.05, 0.01, 0.001, 0.0001 ...
- Enter the formulae:
  - Tap in cell B3 and enter **=sin(A3)**
  - Tap in cell B3 [Edit | Fill | Fill Range] and enter B3:B9 or what is appropriate to match your list of *x*-values
  - In cell C3 enter **=B3/A3** and fill down to C9
  - Tap to open the Cell Viewer window





5. Describe what is happening to the values of *x* and sin *x* as *x* gets closer to 0.

Extend the spreadsheet to determine what happens as x approaches 0 from the negative side.

- Tap in cell A10 or the cell that is beneath your existing list and enter –1
- Enter a series of x-values getting closer to 0 similar to that above
- Extend the formulae in columns B and C

6. Describe what is happening to the values of x and sin x as x approaches 0 from the negative side.

Using Graph

7.

- Open Graph&Table app
- Tap  to display the graph
- Select [Zoom | Quick | Quick Trig] to set an appropriate window

Describe the behaviour of the graph as $x \to 0$.

8. Select [Analysis | Trace] and use the arrow keys to move the cursor along the curve.
What is the largest value for y you see displayed?

9.

Zoom in on the "*y-intercept*":

- Tap [View | Zoom Box]
- Select a region around the *y-intercept*

What is the largest y-value you see displayed using Trace?

10. Repeat zooming in several times and record the largest y-value you see displayed.

11. If there is a precise value for $\lim_{x \to 0} \frac{\sin x}{x}$, what would you predict it to be? Justify your answer.

12. Determine the following limits using either Table, Spreadsheet or Graph:

a) $\lim_{x \to 0} \frac{1 - \cos x}{x}$

b) $\lim_{x \to 0} \frac{\sin 3x}{x}$

c) $\lim_{x \to 0} \frac{\sin x}{\frac{1}{2}x}$

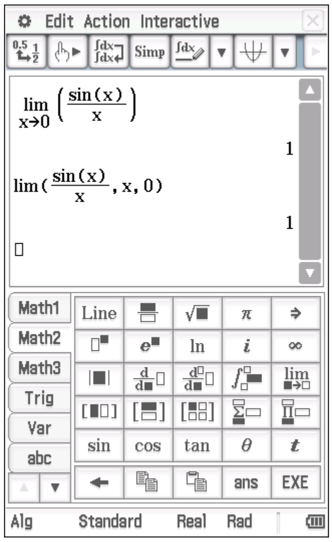
13. Check your answers to Q12 using your CAS.

Calculate limit using CAS:

- In Main, select [lim] from [Math2] in the [Keyboard]
- Complete the entry for desired limit calculation

OR

- In Main
- Select [Action | Calculation | lim]
- Enter the expression of which you wish to determine the limit
- Enter the variable, probably x
- Enter the value where the limit is to be calculated



a) $\lim_{x \to 0} \frac{1 - \cos x}{x}$

b) $\lim_{x \to 0} \frac{\sin 3x}{x}$

c) $\lim_{x \to 0} \frac{\sin x}{\frac{1}{2}x}$

Learning notes

A **formal definition** of a limit:
The $\lim_{x \to a} f(x) = L$ if $f(x)$ can be made arbitrarily close to the limit L just by making x sufficiently close to a.

For continuous functions $\lim_{x \to a} f(x) = f(a)$

In differential calculus, "first principles" use limits to establish the derivative function. The expressions being investigated involve a missing point as the function is undefined at the point of interest. However, providing it is well behaved around that point, we can say there is a limit, and the numerical techniques used in this investigation can often suggest what the limiting value will be.

Using Table, Spreadsheet and Graph, the behaviour of the ratio $\frac{\sin x}{x}$ is explored. From observing this behaviour, you can conjecture the value of the $\lim_{x \to 0} \frac{\sin x}{x}$.

Table and Spreadsheet emphasise the numerical aspect as we look at the numbers resulting from the calculations. Spreadsheets have the advantage of being able to specify the values we use to investigate the behaviour of the function. Graph is a visual representation of the numbers and hence provides a picture of the function's behaviour.
The same techniques can be applied to suggest limits for other expressions.

In each application, the results are suggesting there is a limit for $\frac{\sin x}{x}$ as $x \to 0$. However, it is not yet a formal proof. You now have several approaches enabling you to see what the limit is likely to be. The basis for a geometrical proof of $\lim_{x \to 0} \frac{\sin x}{x}$ is the next investigation. For a formal proof refer to a textbook.

Activity 39 Sine of x on x

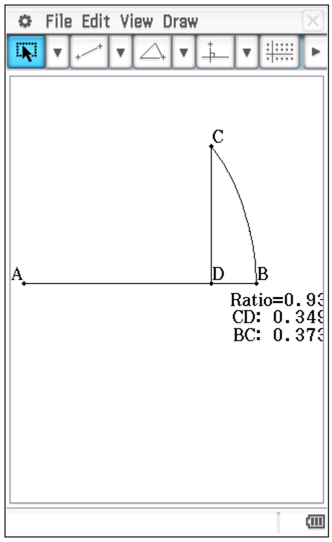
Aim: Establish $\lim_{x \to 0} \frac{\sin x}{x}$ through geometrical arguments

1. Construct the drawing.

(Detailed instructions are in the Learning notes)

- Open Geometry app
- Draw a line segment AB
- Draw an arc BC
- Construct line segment from C to AB
- Set angle ADC to 90
- Set length of AB to 1
- Set angle of AB to 0
- Add measurements of CD and arc BC
- Add an expression to calculate the ratio CD:BC

Dragging C should preserve the properties of the figure.



Drag C closer and closer to B.

a) What do you observe about the arc BC and the perpendicular CD?

b) Begin with angle BAC approximately 30˚ and record measurements as C gets progressively closer to B in the following table.

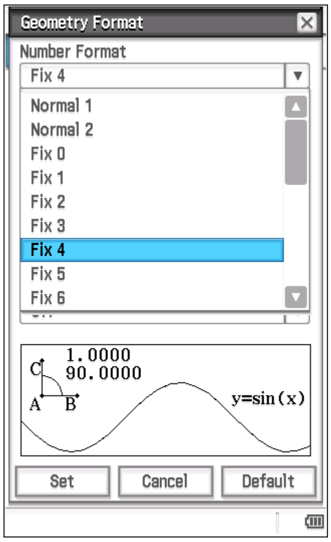
$\overline{CD}$	Arc $\overset{\frown}{BC}$	$\dfrac{\overline{CD}}{\overset{\frown}{BC}}$

Point C approaching point B

- Drag C close to B and record the lengths
- Zoom in, drag C close to B and record the lengths
    - [View | Zoom Box]
    - Tap on one corner of desired zoom area and drag to opposite corner
    - Repeat until the table is complete

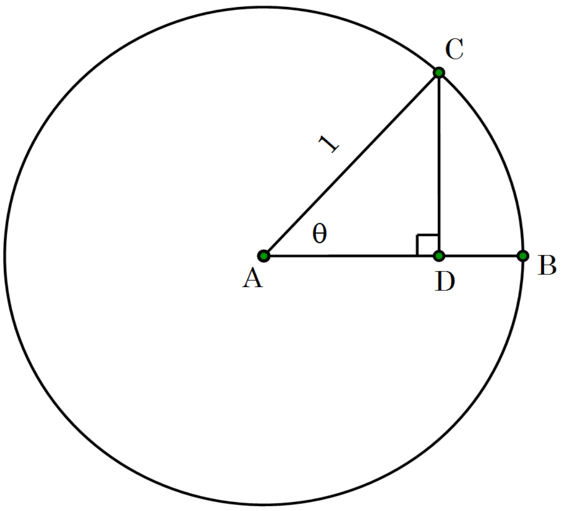
You can change the precision of the measurements displayed on the screen.

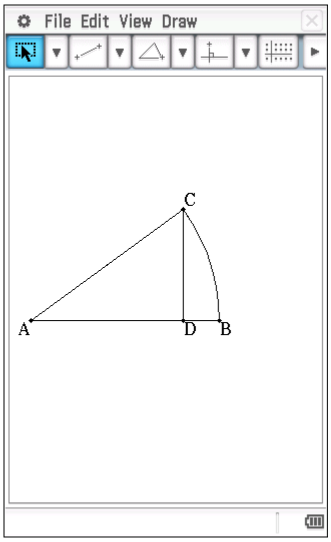
- Tap ⚙
- [Geometry Format | Number Format]
- Choose the number of decimal places you wish to be displayed (Default is Fix 2 for 2 decimal places. 4 decimal places is recommended for this activity)
- Tap Set



c) Look at your table and describe what is happening to the ratio CD:BC as C gets closer to B.

2.





In this diagram, A is the centre of a circle radius 1. B and C are points on the circle, and D is the foot of the perpendicular from C to AB. $\angle BAC = \theta$ (in radians).

a) Calculate in terms of θ:

(i) the length of CD

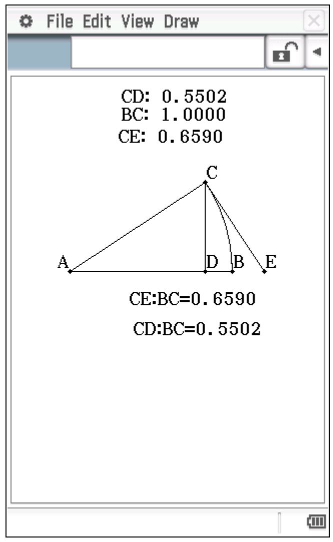
(ii) the arc length BC

(iii) the ratio of CD to BC

b) As CD cannot be longer than CB, there is an upper limit on the ratio $\frac{\overline{CD}}{\overset{\frown}{BC}}$. Express this as an inequality in terms of θ.

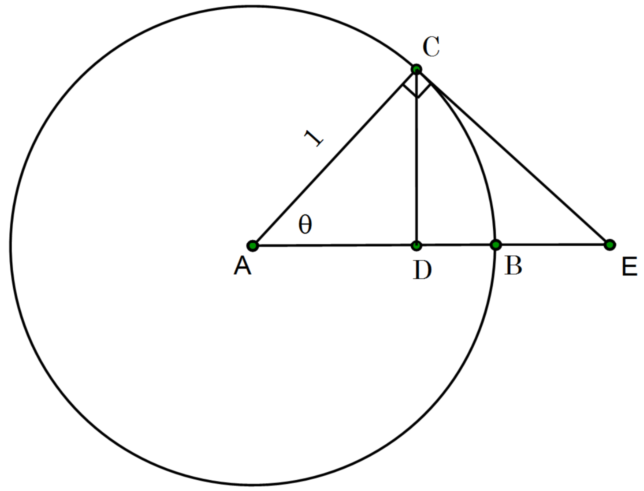
Add a tangent at C to your drawing
(Detailed instructions are part of the Learning notes)

- Construct line segment AC
- Construct line segment from C to AB
- Set angle ACE to 90
- Add measurements for the length of CE and the ratio of CE:BC



3. Explore what happens when you move C closer to B. (Use zoom to enable C to get closer to B.)
What happens to the ratio CE:BC as C approaches B?

4. In this diagram, A is the centre of a circle radius 1.
B and C are points on the circle.
E is the intersection of the tangent at C with the radial line AB.
$\angle BAC = \theta$ (in radians)



Explain why

a) $CE = \tan\theta$

b) $\dfrac{\overline{CE}}{\overset{\frown}{BC}} > 1$

c) $\dfrac{\overline{CE}}{\overset{\frown}{BC}} = \dfrac{\tan\theta}{\theta}$

d) $\dfrac{\sin\theta}{\theta} > \cos\theta$

e) there is a lower limit on the ratio $\dfrac{\sin\theta}{\theta}$, as $\theta \to 0^+$ of 1.

5. Use your findings from questions 2 and 4 to determine the $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta}$.

Learning notes

In this investigation, you are using the Geometry application to find the limit. The proof is based upon the diagrams used and the process of construction and play will help make the reasoning in the proof clearer.

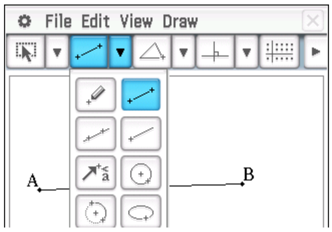
Initially you use measurements taken from the drawing to suggest a value for the $\lim\limits_{x\to 0}\dfrac{\sin x}{x}$. The measurements can be expressed in general terms which enables us to show that there is both a lower limit and an upper limit and that these are identical. Hence, establishing the exact value of the $\lim\limits_{x\to 0}\dfrac{\sin x}{x}$. This is an example of the sandwich theorem.

Construct diagram

- Select Geometry from the Menu
- Select [File | New] to clear the window
- Tap to clear the screen of the integer grid. Cycle through the options until the screen is clear

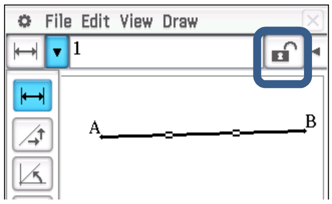
Draw horizontal line segment

- Tap select from the draw pull down menu
- Tap to position one end of the line and tap again for other end



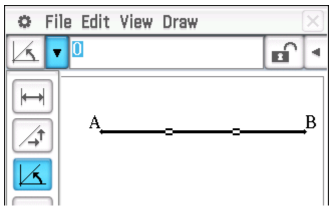
Set to length 1

- Tap to go round the corner
- Tap the line segment
- Select from pull down menu
- Tap in the measurement box and type 1
- Tap to set length



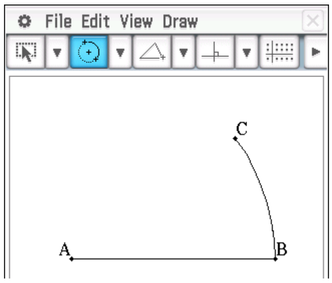
Set direction to horizontal

- Select from pull down menu
- Tap in the measurement box and type 0
- Tap to set length



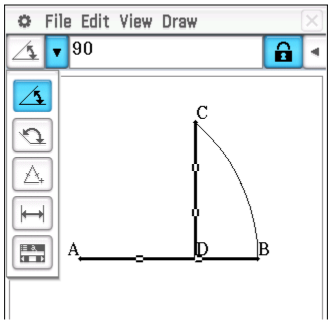
Draw arc

- Tap to go back round the corner
- Select from the draw pull-down menu
- Tap on A
- Tap on point B
- Tap on another point on the circle that has appeared



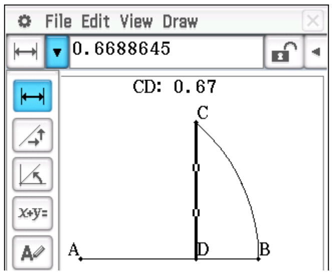
Draw perpendicular

- Tap to select line segment tool
- Tap point C
- Tap on line segment AB
- Tap to go round the corner
- Tap the line segments AB and CD
- Select from the pull-down measure menu
- Tap in measurement box
- Enter 90
- Tap



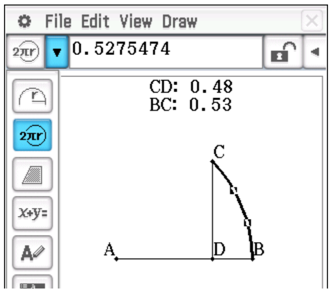
Measure length of perpendicular CD

- Tap in open space
- Tap CD to select line segment
- Open pull-down measure menu
- Select length
- Tap on in the top menu
  The length should now be displayed on the screen and you can edit the descriptor
- Tap



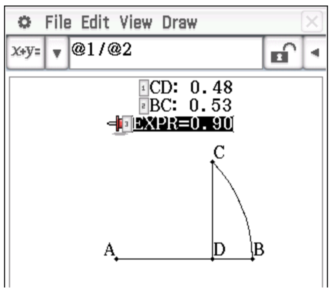
Measure length of arc BC

- Tap in open space
- Tap arc BC to select the arc
- Open pull-down measure menu
- Select arc length
- Tap on in the top menu
  The length should now be displayed on the screen and you can edit the descriptor
- Tap



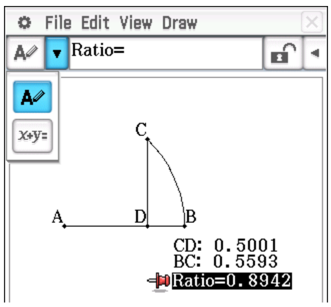
Calculate ratio

- Tap in open space
- Select [Draw | Expression]
  Numbers will appear in front of the on-screen measurements
- Tap on the number in front of the length CD
- Press ÷
- Tap on the number in front of the arc length BC
- Tap to complete the expression @1/@2



Change label of expression

- Tap in open space
- Tap on the value of the expression
- Select
- Change label in Measurement box
- Tap in open space
- Select all three labels and move to just below point B
  This will keep them visible when you zoom in



Drag C progressively closer to B

- Select tool
- Tap in open space
- Tap on point C
- Tap on C and drag to new position

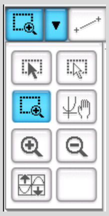
Zoom in:

- Select or [View | Zoom Box]
- Tap on one corner of desired zoom area and drag to opposite corner

You may wish to then:

- Tap or [View | Select]
- Tap in open space, tap on the expression and drag to keep visible for next zoom

View menu icons



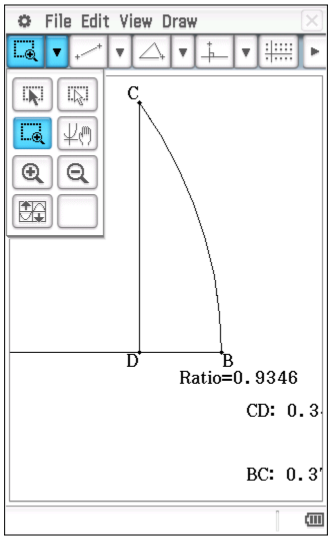
select tool

zoom box

pan across the window (move drawing)

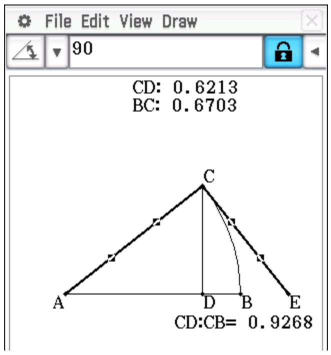
zoom in

zoom out



Add a tangent at C to your drawing

- Draw AC
  - Tap
  - Tap on A, tap C
- Draw CE
  - Tap C, tap AB
- Tap ▶ to go round the corner
- Select lines AB and CE
- Set angle to 90

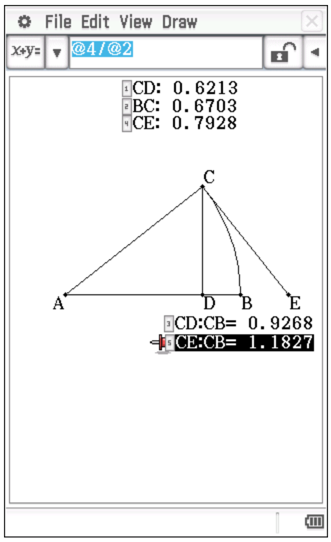


Add measurement for the length of CE

- Tap in open space
- Tap on line CE
- Select [Draw | Measurement | Length].

Calculate the ratio of CE:BC

- Select [Draw | Expression]
- Tap in the measurement box
- Press the Keyboard button
- Enter the expression to be calculated
  If CB is measurement 2 and CE is measurement 4, the expression is @4/@2
- Tap EXE or tap



Activity 40 Modelling motion

Aim: Model motion along a straight line

Ball toss

1. Mitch throws a cricket ball straight up in the air. Peter records the throw on his iPad to get the following data on the height of the ball.

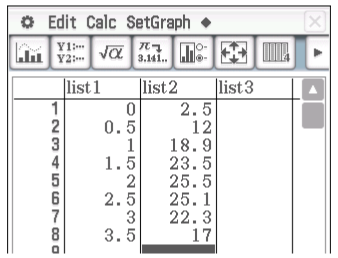
Time	0	0.5	1.0	1.5	2	2.5	3	3.5
Height	2.5	12	18.9	23.5	25.5	25.1	22.3	17

Time is in seconds and height in metres.

a) Model this data to obtain a height function.

Calculate model using regression

- Enter the data into Statistics.
- Draw a scatter graph.
- Use the regression that fits the shape of your graph.
- You may like to save the function to y1 so you can work with unrounded values.



Use your model to determine:

b) the velocity function

c) when the velocity is 0

d) the acceleration function

e) the maximum velocity in the interval 57$0 \le t \le 4.457$

f) the maximum speed in the interval $0 \le t \le 4.457$

2. Physics students will have used this equation: $s = ut + \frac{1}{2}at^2$, or something similar, for describing motion under constant acceleration. Note that s is a function of t and a and u are constants.

a) Calculate the velocity function.

b) Calculate the acceleration function.

Air hockey

What if the acceleration is not uniform?

3. Tom is playing air hockey. He moves his slider back and forth along a straight line. Measurements of the position of the slider over a short period of play are recorded in the table below.

Time (t) seconds	0	0.5	1.0	1.5	2	2.5
Distance from edge of table (x) cm	10	30	28	43	56	40

a) Model the paddle's distance from the edge of the table (x) as a function of time (t) with a quartic function.

According to your model, determine when the:

b) paddle is stationary

c) paddle is moving fastest

d) paddle is moving to the left

e) when the paddle's acceleration is

(i) least

(ii) 0

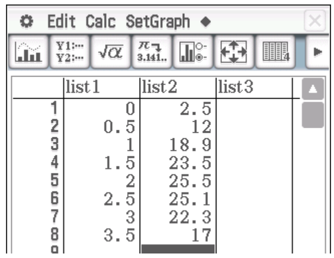
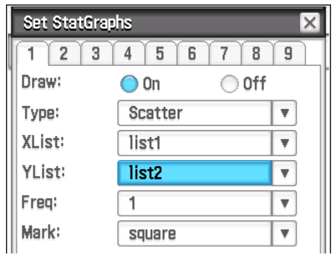
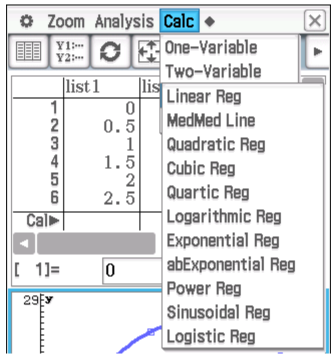
(iii) greatest

Learning notes

Models are an important part of mathematics. We can use experimental data to generate a model and then use it to make predictions. Clearly if the predictions aren't appropriate we will revise the model.

To model data:

- Enter the data into Statistics.
- Draw the graph.
- Choose a regression that fits the shape of the data.

Enter the data into Statistics • Open Statistics app • Select [Edit ∣ Clear All] • Enter the data, time in list1 and distance or position in list2	
Draw a scatter graph • Tap [icon] • Ensure settings for StatGraph 1 are as shown • Tap Set • Tap [icon]	
Calculate regression • Use the regression that fits the shape of your graph Select [Calc ∣ … Reg] • You may like to save the function e.g. to y1 You can then work with this function in Main and it will use unrounded values	

Solutions

Activity 1 A function equal to its gradient

1.

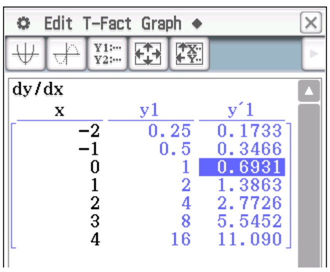
a)

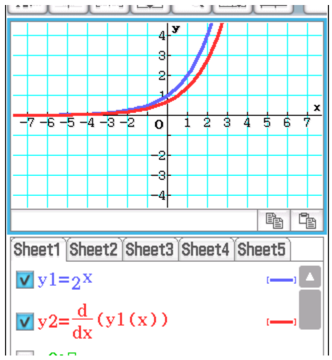
x	–2	–1	0	1	2	3	4
$y = 2^x$	0.25	0.5	1	2	4	8	16
$\frac{dy}{dx}$	0.1733	0.3466	0.6931	1.386	2.773	5.545	11.09

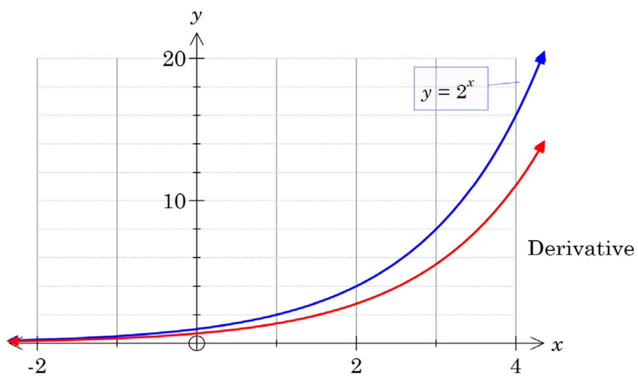
b) The derivative values are all smaller than their respective y-values.

2.

a)







b) The derivative is a similar shape but lower than the function. It could be a vertical dilation.

c) The graphs are of a similar shape but closer together and the derivative is above the curve $y = 3^x$.

d) Somewhere between 2.5 and 2.9.

3. Answers will vary. This is an example

a	Function $y = a^x$	Equation of derivative (using abExponential regression)
2	$y = 2^x$	$y = 0.6931 \times 2^x$
3	$y = 3^x$	$y = 1.0896 \times 3^x$
2.5	$y = 2.5^x$	$y = 0.9162 \times 2.5^x$
2.7	$y = 2.7^x$	$y = 0.9932 \times 2.7^x$
2.8	$y = 2.8^x$	$y = 1.0296 \times 2.8^x$
2.72	$y = 2.72^x$	$y = 1.0006 \times 2.72^x$

4.

a) $f(x) = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \ldots$ It has one less term but this is not relevant for an infinite series.

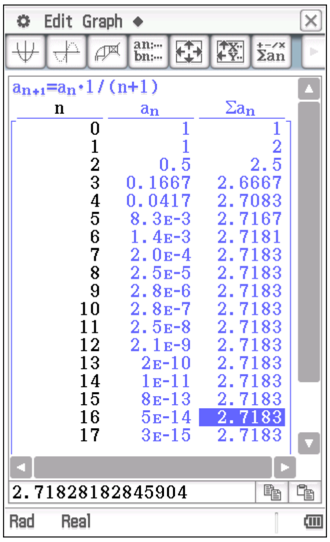
b)

$$f(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = 2.71\dot{6}$$

$$f'(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = 2.708\dot{3}$$

c) 16 terms

d) 2.71828182845904



e) There is only one graph, i.e. they are the same.

f) $y = 2.7182^x$

Activity 2 Differentiating exponential functions

1.

Working	Justification
$f'(x)=\lim_{h\to 0}\dfrac{a^{x+h}-a^x}{h}$	First principles definition of derivative
$=\lim_{h\to 0}\dfrac{a^x.a^h-a^x}{h}$	Index law $a^m\times a^n=a^{m+n}$
$=a^x\lim_{h\to 0}\dfrac{a^h-1}{h}$	Common factor a^x; can be removed from the limit as it is independent of h

2. a) $\left.\dfrac{a^h-1}{h}\right\|_{h=0}=\dfrac{0}{0}$ i.e. undefined.

b)

a	Estimate of $\lim_{h\to 0}\dfrac{a^h-1}{h}$
2	0.693
3	1.10
4	1.39
2.6	0.956
2.7	0.993
2.8	1.03
2.71828	1.00

c) Given $\dfrac{d}{dx}\left(a^x\right)=a^x\lim_{h\to 0}\dfrac{a^h-1}{h}$ when $a=2.71828$, $\lim_{h\to 0}\dfrac{a^h-1}{h}\approx 1$, hence

$\dfrac{d}{dx}\left(2.71828^x\right)\approx 2.71828^x$

3. a) (i) 1.0986

(ii) 0.99325

(iii) 0.99990

b) (i) $a\approx 2.718281828$

(ii) $a=e$

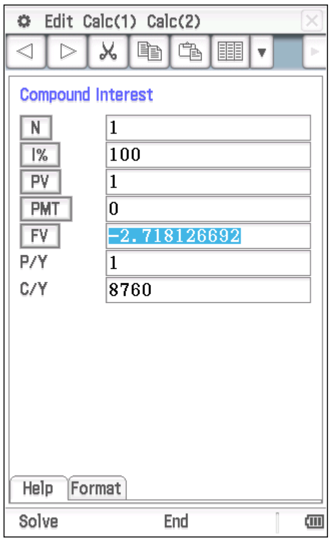
c) $a=e$, i.e. $\dfrac{d}{dx}\left(e^x\right)=e^x$

Activity 3 Route 2.7…e

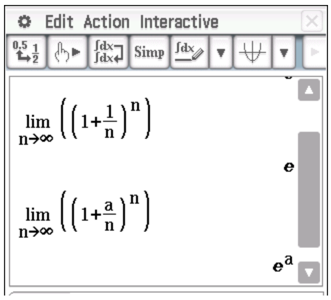
1. After 1 year, value of investment is \$11 000.
2.

$$\begin{aligned}\text{Value} &= 10\ 000 \times 1.05^2 \\ &= \$11\ 025\end{aligned}$$

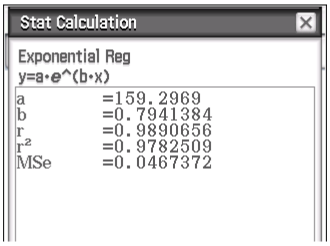
3.
   a) \$11 038.13
   b) \$11 047.13
   c) \$11 050.65
   d) \$11 051.56
   e) \$11 051.70
4.
   a) \$2
   b) \$2.25
   c) \$2.44140625
   d) \$2.61303529
   e) \$2.692596954
   f) \$2.714567482
   g) \$2.718126692



5. 2.718281828
   The limited number of decimal places on the calculator suggests that the constant may be a recurring decimal. In fact it is not. To 20 decimal places, the constant is
   $e \approx$ 2.71828 18284 59045 23536. . .
6. a) e
   b) e^a



Activity 4 Growth and decay

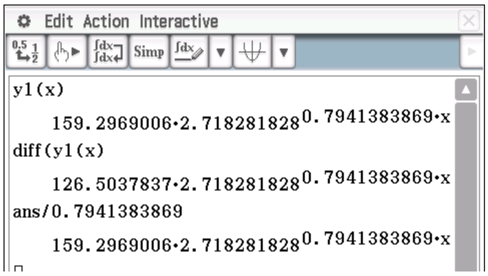


1. $P \approx 159\, e^{0.794t}$ where t is the number of years since 2007.

2. As shown below, the original function is a scalar multiple of the derivative. We see that

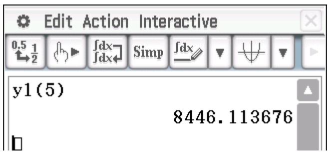
$$\frac{dP}{dt} \approx 0.794 \times 159e^{0.794t}$$

i.e. $\frac{dP}{dt} \approx kP \quad k = 0.794$



3. The instantaneous rate of increase in the number of notifications of whooping cough is proportional to the number of existing notifications.
The k value represents this proportionality constant, that is, at any time, the rate of growth of the number of notifications is approximately 79% of the number of notifications at that time per year.

4. 2012 corresponds to $t = 5$.
Number of notifications ≈ 8450



5. During the year 2014 ($t \approx 6.1$) the number of notifications is predicted to reach 20 000.

6. Many factors affect the mathematical model in the future, for example, if the vaccine for parents is successful, the number of cases should be significantly lower than the model predicts. Other limitations might include the limiting factor of the total population of WA – clearly the number of cases cannot exceed this figure.

7. $$A = A_0\, e^{\lambda t}$$

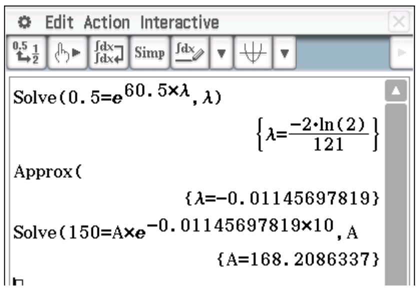
let $\frac{1}{2} A_0 = A_0\, e^{\lambda \times 60.5}$

$$\lambda \approx -0.011457$$

Now, when $t = 10$ days, A = 150g.

$$150 = A_0\, e^{-0.011457 \times 10}$$

$$A_0 \approx 168.2\text{g}$$



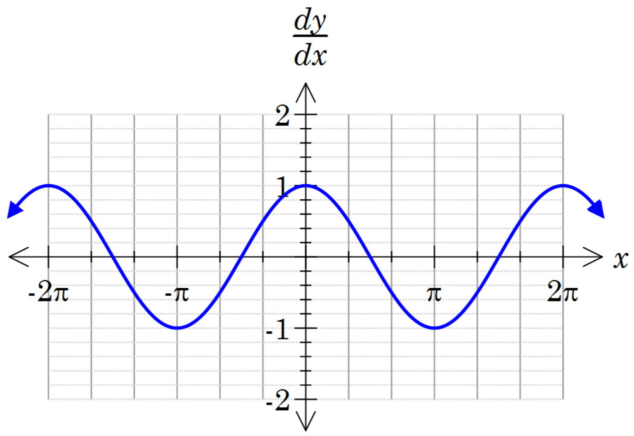
Activity 5 Differentiating trigonometric functions

1. a) Since the graph of $y = \sin(x)$ is periodic, it makes sense that the gradient function graph is also periodic.

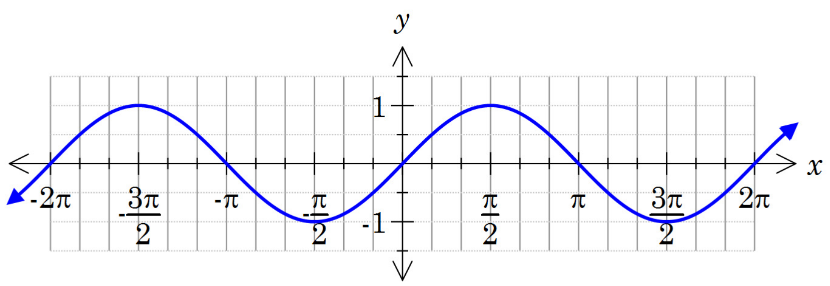
   b) The period is likely to be 2π (the same as the function graph).

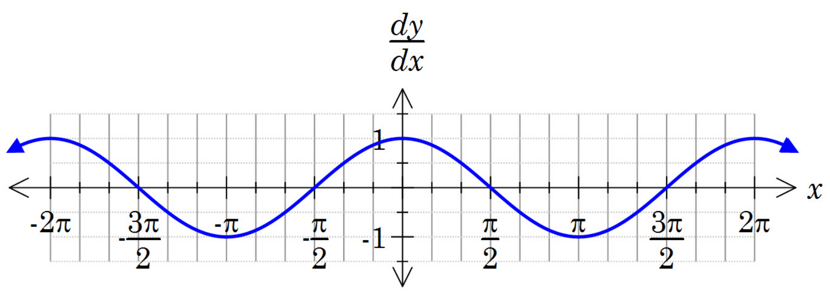
   c) $x = \dfrac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$

   d)



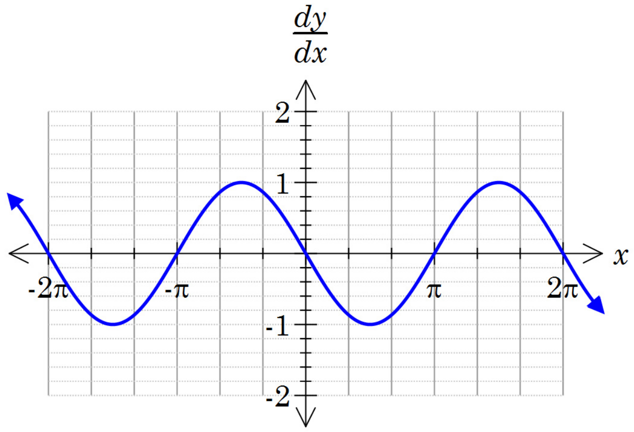
2. a)





   a) $\dfrac{dy}{dx} = \cos(x)$

3.



4.

Working	Justification
$f'(x)=\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$	First principles definition of derivative
$=\lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}$	Compound angle formula for $\sin(A+B)$
$=\lim_{h\to 0}\frac{\sin(x)\cos(h)-\sin(x)}{h}+\lim_{h\to 0}\frac{\cos(x)\sin(h)}{h}$	Grouping terms into two separate limits
$=\sin(x)\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}$	Factoring out $\sin(x)$ and $\cos(x)$ since they are independent of h
$=\cos(x)$	$\lim_{h\to 0}\frac{\sin(h)}{h}=1$ and $\lim_{h\to 0}\frac{\cos(h)-1}{h}=0$

5.

$$f'(x)=\lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h}$$

$$=\lim_{h\to 0}\frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h}$$

$$=\lim_{h\to 0}\frac{\cos(x)\cos(h)-\cos(x)}{h}-\lim_{h\to 0}\frac{\sin(x)\sin(h)}{h}$$

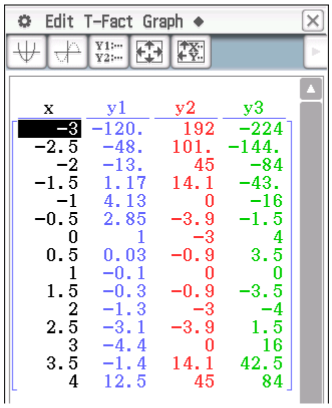
$$=\cos(x)\lim_{h\to 0}\frac{\cos(h)-1}{h}-\sin(x)\lim_{h\to 0}\frac{\sin(h)}{h}$$

$$=-\sin(x)$$

Activity 6 The second derivative

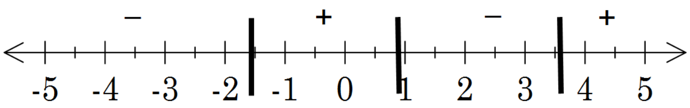
1. a)

x	$f(x)$	$f'(x)$	$f''(x)$
–1.57	0	17.2	–47.3
–1	4.13	0	–16
–0.41	2.49	–4	0
0.53	0	–0.8	3.34
1	–0.1	0	0
2.41	–2.8	–4	0
3	–4.4	0	16
3.59	0	18.2	48.8

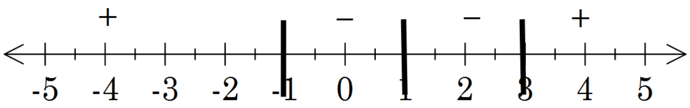


x	y1	y2	y3
−3	−120.	192	−224
−2.5	−48.	101.	−144.
−2	−13.	45	−84
−1.5	1.17	14.1	−43.
−1	4.13	0	−16
−0.5	2.85	−3.9	−1.5
0	1	−3	4
0.5	0.03	−0.9	3.5
1	−0.1	0	0
1.5	−0.3	−0.9	−3.5
2	−1.3	−3	−4
2.5	−3.1	−3.9	1.5
3	−4.4	0	16
3.5	−1.4	14.1	42.5
4	12.5	45	84

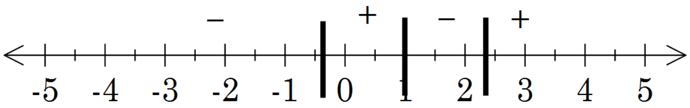
b) $f(x)$



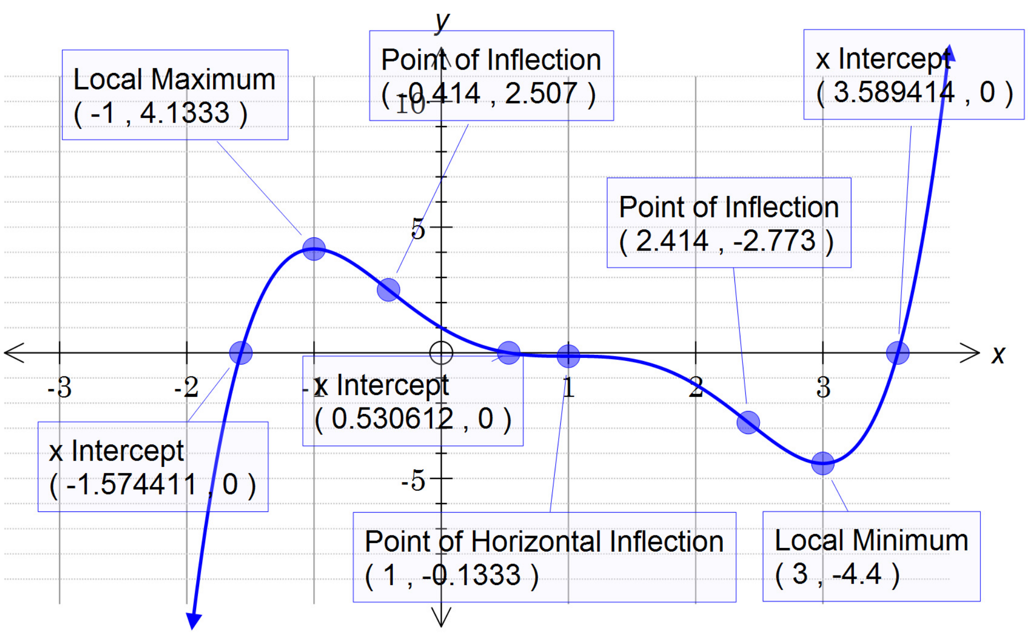
c) $f'(x)$



d) $f''(x)$



e)



2.

a) $f'(x) > 0$ the function is increasing/~~decreasing/indeterminate~~.

b) $f'(x) = 0$ the function/~~gradient~~ is ~~increasing/decreasing~~/indeterminate.

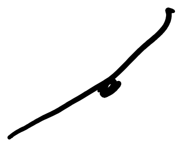
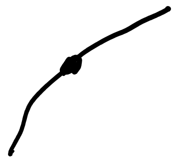
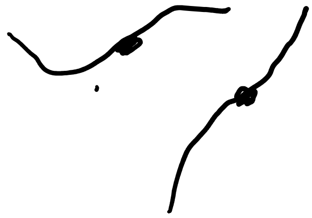
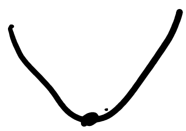
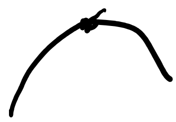
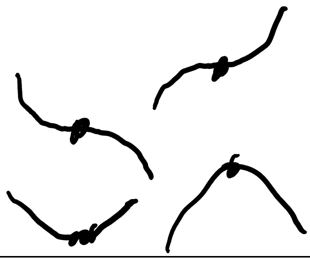
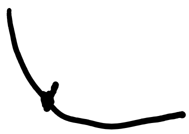
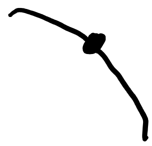
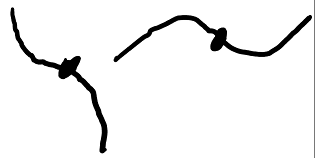
c) $f'(x) < 0$ the function/~~gradient~~ is ~~increasing~~/decreasing/~~indeterminate~~.

d) $f''(x) > 0$ the ~~function~~/gradient is increasing/~~decreasing/indeterminate~~.

e) $f''(x) < 0$ the ~~function~~/gradient is ~~increasing~~/decreasing/~~ indeterminate~~.

f) $f''(x) = 0$ the ~~function~~/gradient is ~~increasing/decreasing~~/ indeterminate.

3.

	$f''(x) > 0$	$f''(x) < 0$	$f''(x) = 0$
$f'(x) > 0$			
$f'(x) = 0$			
$f'(x) < 0$			

1.

b)

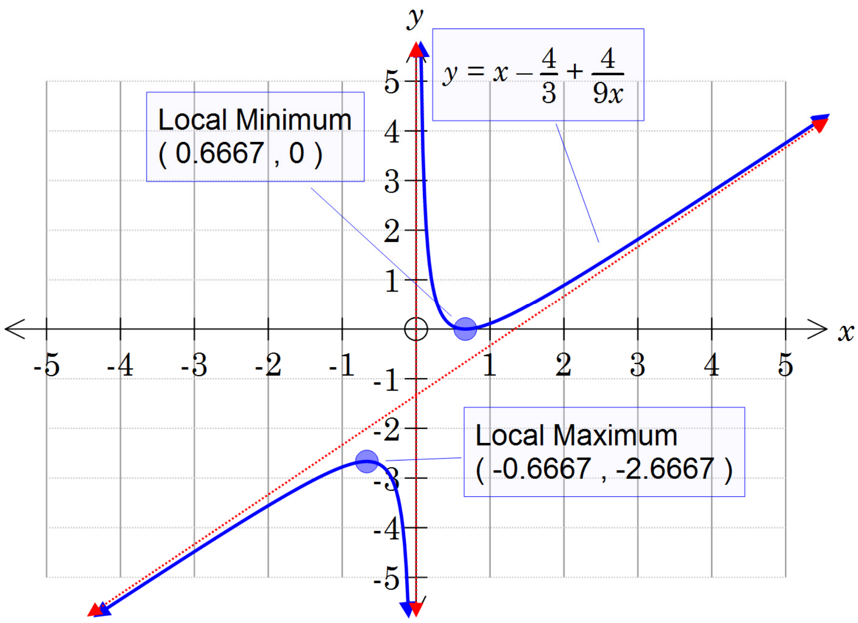
(i) Determine the derivative function.

(ii) Solve for the derivative =0 to locate stationary points.

(iii) Calculate second derivative at stationary points to determine nature of stationary points. If the second derivative is negative it means the gradient is decreasing hence a local maximum. If positive then a local minimum.

(iv) Calculate *y*-coordinates of stationary points.

c)

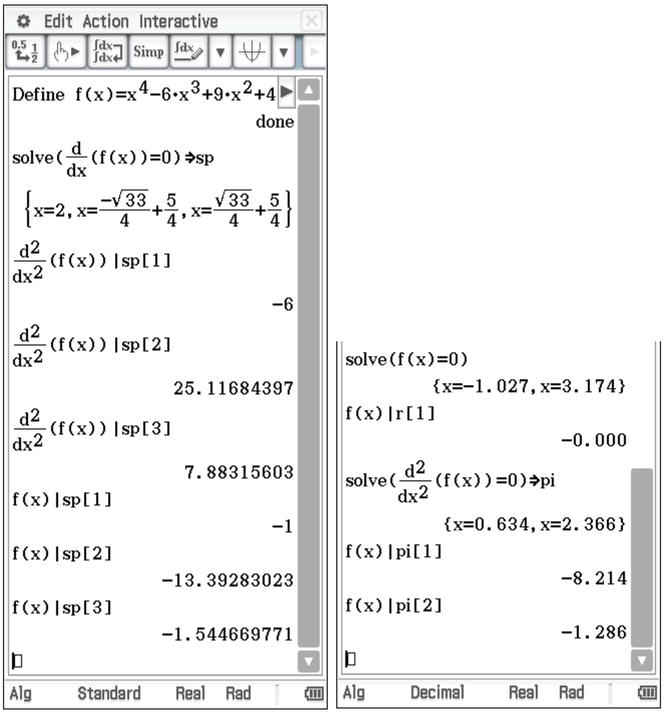


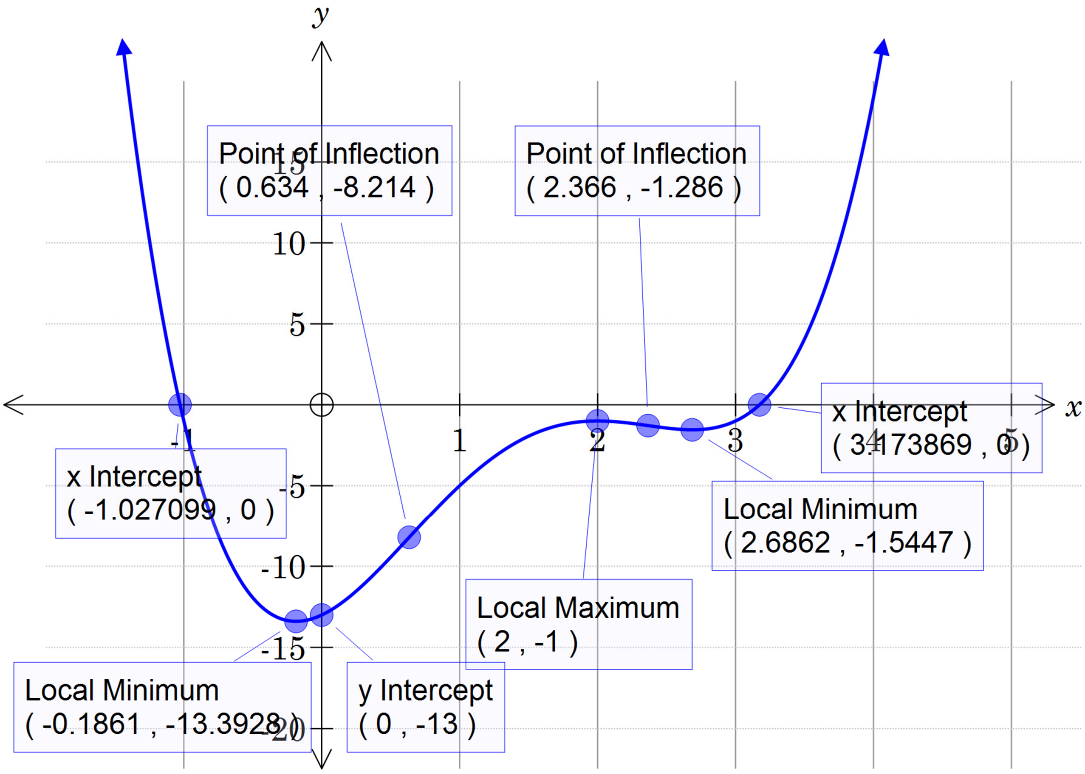
Vertical asymptote at $x = 0$.

As $\begin{array}{l} x \to \infty, \quad y \to x - \frac{4}{3} \text{ from above} \\ x \to -\infty, \quad y \to x - \frac{4}{3} \text{ from below} \end{array}$.

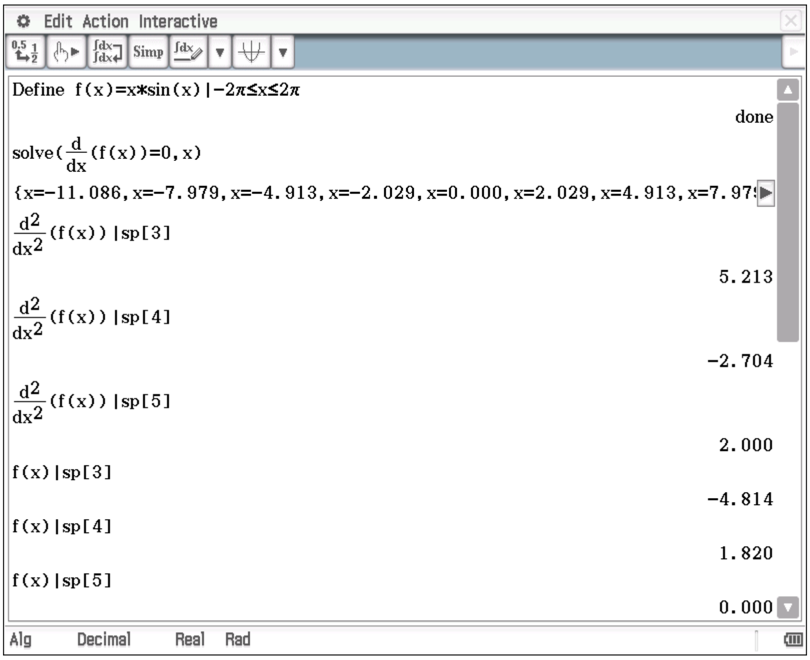
The line $y = x - \frac{4}{3}$ is known as an oblique asymptote.

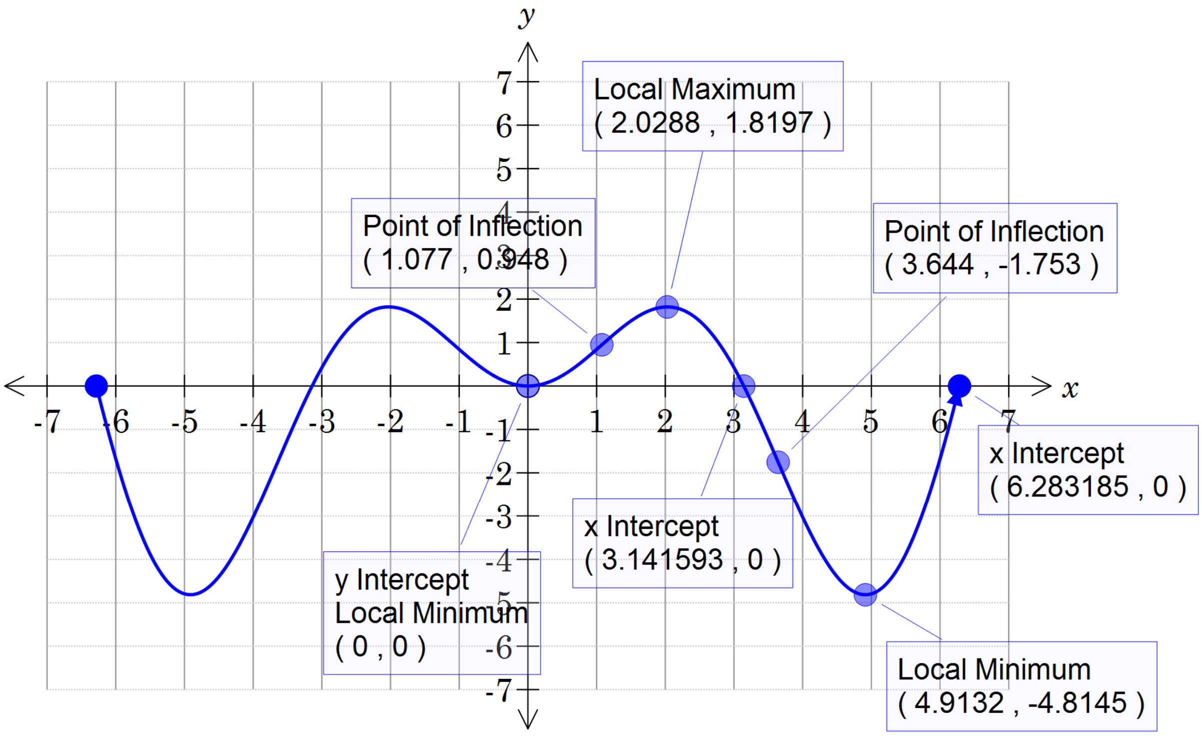
2. a) CAS working





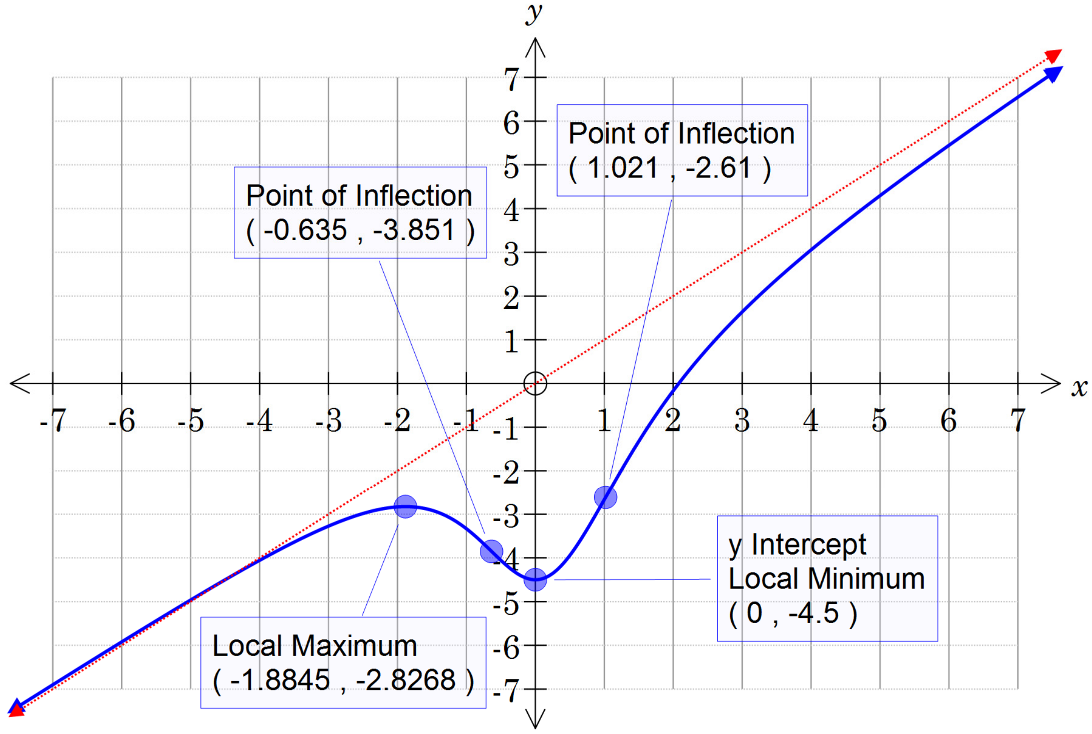
b)





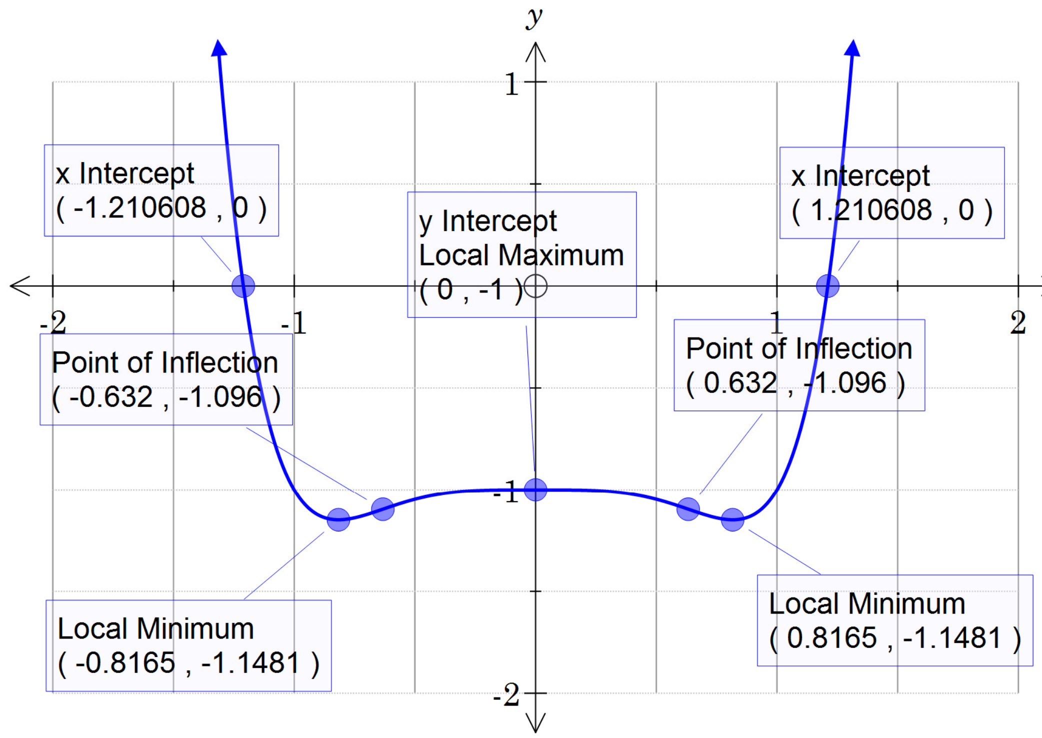
Note: graph is symmetrical about y-axis.

c) As $x \to \infty,\ \ y \to \infty\ \ (y \to x^{-})$
As $x \to -\infty,\ \ y \to 0 - \infty\ \ (y \to x^{+})$



This graph is interesting in that there is an oblique asymptote of $y = x$ and the curve passes through the asymptote at (–4.5, –4.5) and as $x \to -\infty$ the curve approaches the asymptote from above.

d) At $x = 0$ both first and second derivatives are 0.



Activity 8 Composite functions

1. a) (i) $g(3) = 2$

   (ii) $f(2) = -1$

   (iii) $f(g(3)) = -1$

   b) $f(g(3))$ is the same as $f(2)$ since $g(3) = 2$.

2. $f(g(x)) = \dfrac{1}{\sqrt{x+1}-3}$

3. a) $g(f(x)) = \sqrt{\dfrac{x-2}{x-3}}$

   b) Expected $g(f(x)) = \sqrt{\dfrac{1}{x-3}+1}$

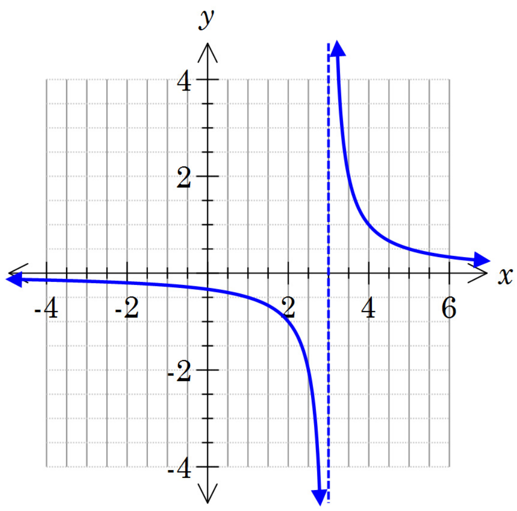
   These are equivalent:

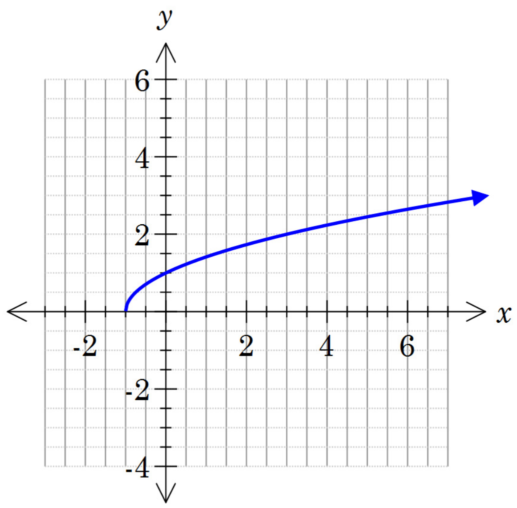
   $\sqrt{\dfrac{1}{x-3}+1}$

   $= \sqrt{\dfrac{1}{x-3}+\dfrac{x-3}{x-3}}$

   $= \sqrt{\dfrac{x-2}{x-3}}$

4. a)





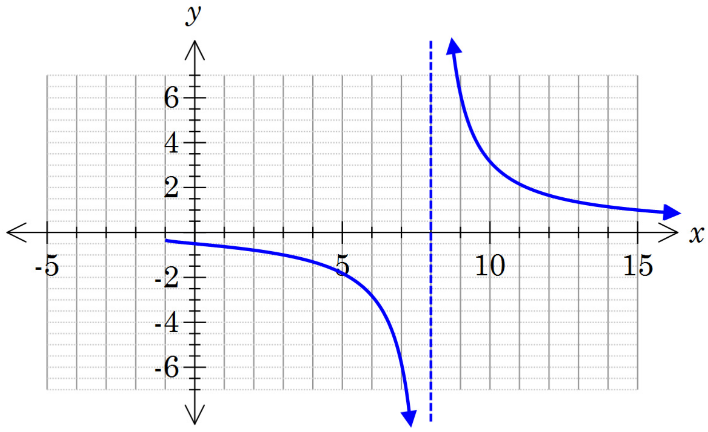
b) $f(x) = \dfrac{1}{x-3}$

Domain: $x \in \mathbb{R}, x \neq 3$
Range: $y \in \mathbb{R}, y \neq 0$

$g(x) = \sqrt{x+1}$

Domain: $x \in \mathbb{R}, x \geq -1$
Range: $y \in \mathbb{R}, y \geq 0$

5.

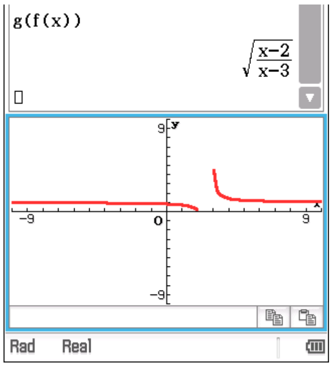


$$f(g(x)) = \frac{1}{\sqrt{x+1} - 3}$$

Domain:
$x \in \mathbb{R}, x \geq -1, x \neq 8$

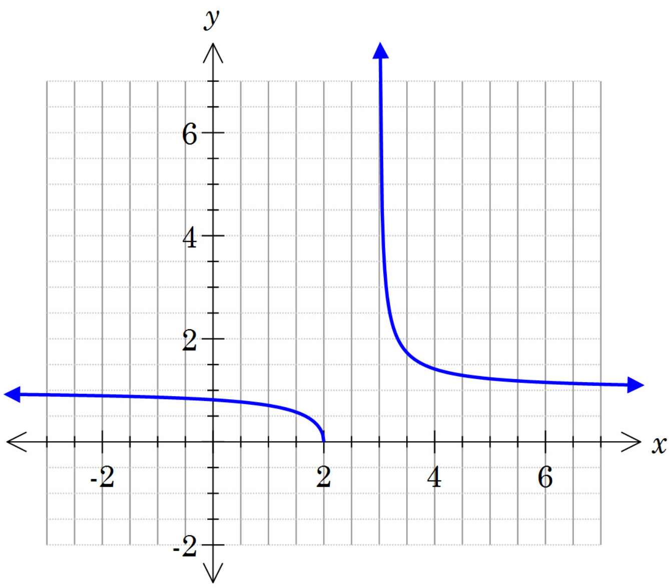
Range:
$y \in \mathbb{R}, y > 0, y \leq -\frac{1}{3}$

6. a)



b) Whilst the graph appears to terminate as $x \to 3^+$, this is not the case. The graph has a vertical asymptote at $x = 3$.

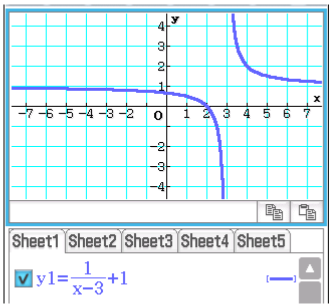
c)



Domain: $x \in \mathbb{R}, x \leq 2, x > 3$

Range: $y \in \mathbb{R}, y > 0, y \neq 1$

d) This function clearly shows the vertical asymptote at $x = 3$. The graph in a) has a restricted domain (as a result of the square root) and is missing the part between $x = 2$ and $x = 3$.



Activity 9 Gradient of composite functions

1.

a) $x^6 - 9x^5 + 30x^4 - 45x^3 + 30x^2 - 9x + 1$

b) $6x^5 - 45x^4 + 120x^3 - 135x^2 + 60x - 9$

c) $3\left(x^2 - 3x + 1\right)^2 \cdot (2x - 3)$

d) 3 from the cube
The quadratic squared, i.e. one degree less than a cube.
The derivative of the quadratic

e) When differentiating a composite function: First differentiate the outer function and then multiply by the derivative of the inner function.

2. Answers will vary

3.

a) $\left(x^2 + 6x - 1\right)^4$

b) e^{3x^2+5x}

c) $\sin(x^3 - 7)$

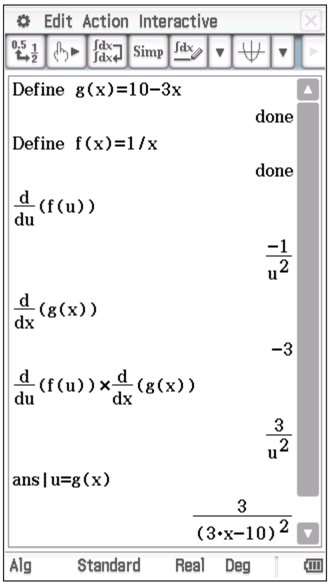
d) $\left(3x + 4\right)^{-2}$

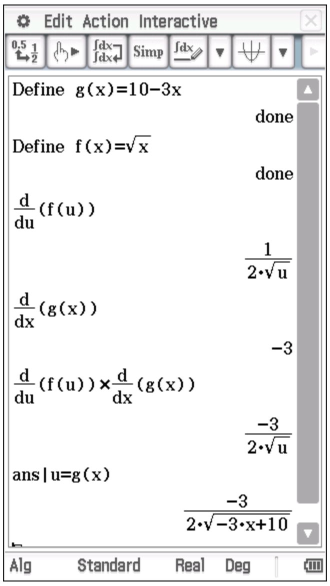
e) $\left(x^2 - 4\right)^{\frac{1}{2}}$

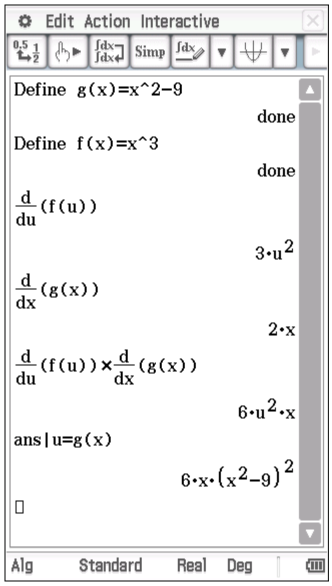
f) $-4\cos(x^2 + 6x - 1)$

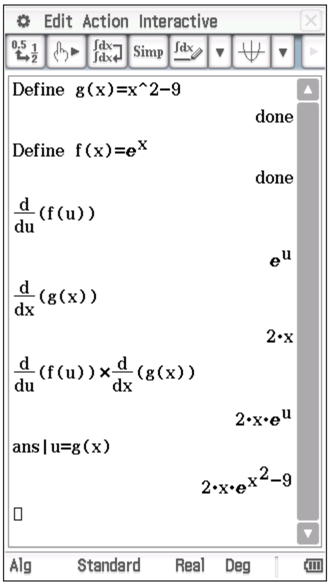
4.

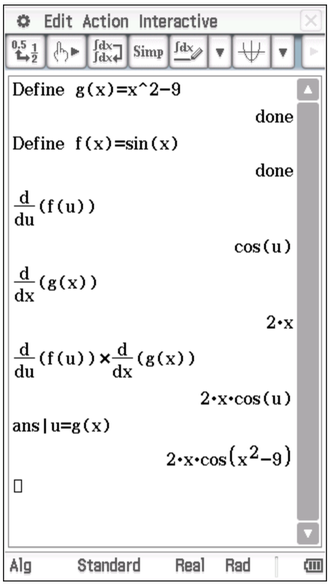
Composite function $y = f(g(x))$	Inner function $u = g(x)$	Outer function $f(u)$	$\frac{dy}{du}$	$\frac{du}{dx}$	$\frac{dy}{du} \times \frac{du}{dx}$	$\frac{dy}{dx}$
$\frac{1}{10 - 3x}$	$10 - 3x$	u^{-1}	$-\frac{1}{u^2}$	-3	$\frac{3}{u^2}$	$\frac{3}{\left(10 - 3x\right)^2}$
$\sqrt{10 - 3x}$	$10 - 3x$	$u^{\frac{1}{2}}$	$\frac{1}{2}u^{-\frac{1}{2}}$	-3	$-\frac{3}{2}u^{-\frac{1}{2}}$	$-\frac{3}{2}\left(10 - 3x\right)^{-\frac{1}{2}}$
$\left(x^2 - 9\right)^3$	$x^2 - 9$	u^3	$3u^2$	$2x$	$3u^2 \times 2x$	$3\left(x^2 - 9\right)^2 \times 2x$
e^{x^2-9}	$x^2 - 9$	e^u	e^u	$2x$	$e^u \times 2x$	$e^{x^2-9} \times 2x$
$\sin(x^2 - 9)$	$x^2 - 9$	$\sin(u)$	$\cos(u)$	$2x$	$\cos(u) \times 2x$	$\cos(x^2 - 9) \times 2x$
$e^{3\sin x}$	$3\sin x$	e^u	e^u	$3\cos x$	$e^u \times 3\cos x$	$e^{3\sin x} \times 3\cos x$

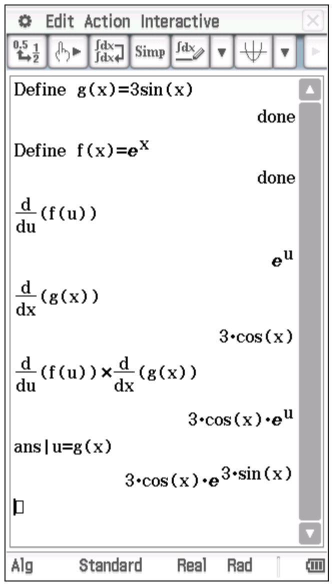












5.

$$\frac{d}{dx}\sqrt{10-\frac{3}{x^2-9}} = \frac{1}{2}\left(10-\frac{3}{x^2-9}\right)^{-\frac{1}{2}} \times \frac{d}{dx}\left(10-\frac{3}{x^2-9}\right)$$

$$= \frac{1}{2}\left(10-\frac{3}{x^2-9}\right)^{-\frac{1}{2}} \times \left(3\left(x^2-9\right)^{-2}\right) \times \frac{d}{dx}\left(x^2-9\right)$$

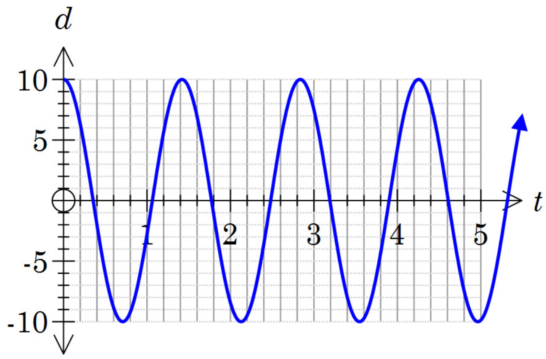
$$= \frac{1}{2}\left(10-\frac{3}{x^2-9}\right)^{-\frac{1}{2}} \times \left(3\left(x^2-9\right)^{-2}\right) \times 2x$$

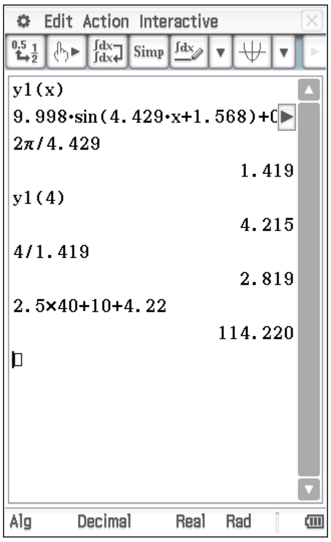
$$= \frac{3x}{\left(x^2-9\right)^2\sqrt{10-\frac{3}{x^2-9}}}$$

Activity 10 Pendulum motion

1. a) $d = 10.0\sin(4.43t + 1.57)$

(The vertical translation has been omitted given its insignificance and inconsistency with the situation)





b)

$$T \approx \frac{2\pi}{4.43}$$

$$\approx 1.42 \text{ s}$$

c) 4.2 cm

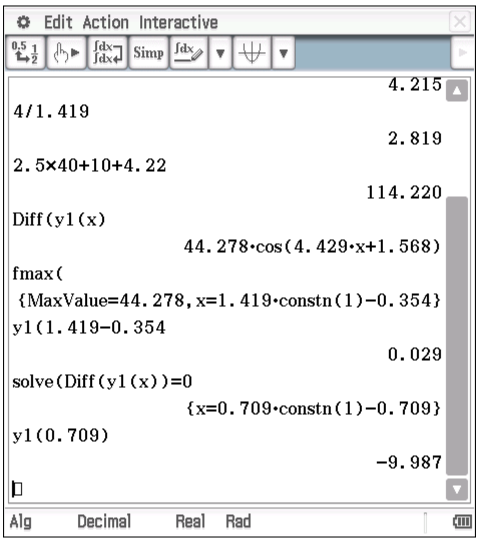
d) The bob has completed ~2.82 cycles.

Each cycle is $4 \times 10 = 40$ cm

Hence total distance $= 2.5 \times 40 + 10 + 4.22$

$= 114.22$ cm

2.



a) $v = 44.3\cos(4.43t + 1.57)$

b) Max velocity 44.3 cm/s

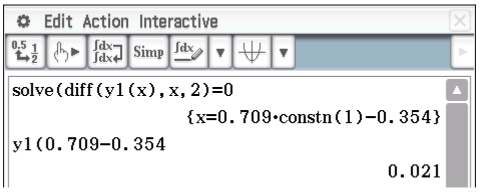
Occurs at $t \approx 1.07$ s

(Max speed first occurs when $t \approx 0.36$ s in a negative direction)

c) Max velocity occurs when bob is at equilibrium position. Error in screenshot is due to rounding.

d) Bob is stationary ($v = 0$) when $d = -10$ and when $d = 10$ cm

3. Acceleration is zero at equilibrium position $d = 0$. Error in screenshot is due to rounding.



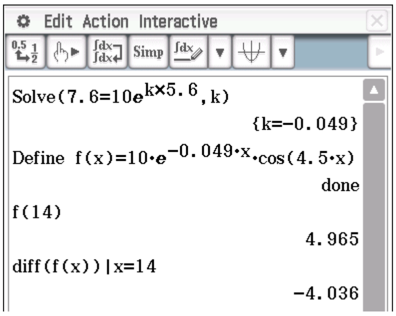
4. a) (i) The bob starts at maximum displacement. A sine function would require a phase shift.

(ii) 10 cm is the initial maximum displacement.

(iii) With period $T \approx 1.4$ s, $b \approx \dfrac{2\pi}{1.4} \approx 4.5$

b) $k \approx -0.049,\ d = 10e^{-0.049t}\cos 4.5t$

c) $t = 14$ s, $d \approx 5.0$ cm, $v \approx -4.0$ cm/s (4 cm/s to the left)



Activity 11 Comfy chairs

1.

a) $\{x \mid 0 < x \le 250\}$

b) Profit = Revenue – Costs

$$P = \left(240 + \frac{600}{x}\right)x - \left(2400 + 42x + 12x^{\frac{3}{2}}\right)$$

$$= 198x - 1800 - 12x^{\frac{3}{2}}$$

c) Max when $\frac{dP}{dx} = 0$ or at end points

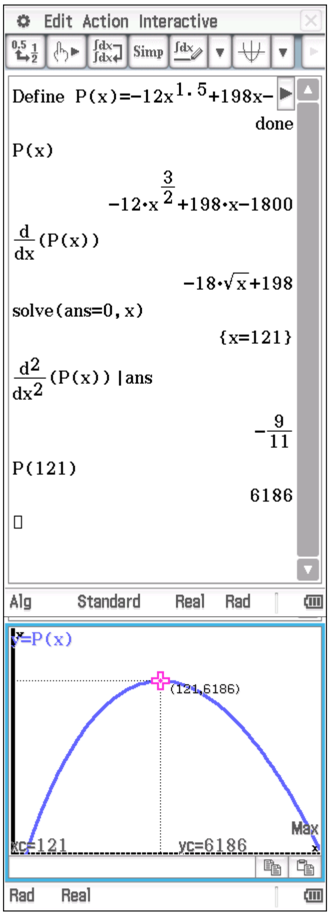
$$\frac{dP}{dx} = -18\sqrt{x} + 198$$

$$\frac{dP}{dx} = 0 \Rightarrow x = 121$$

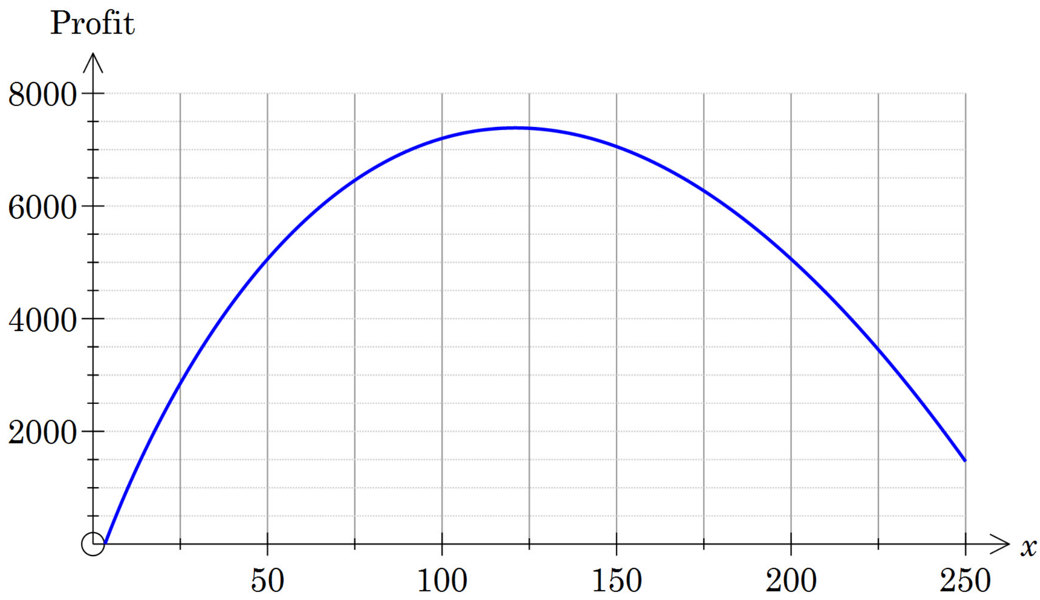
$$\left.\frac{d^2P}{dx^2}\right|_{x=121} < 0$$

∴ Local max when $x=121$

d) $P(121) = 6186$ Max profit is $6186 when 121 chairs are manufactured



e)



2.

Profit = Revenue –Costs

$$P = \left(325 + \frac{600}{x}\right)x - \left(2400 + 55x + 12x^{\frac{3}{2}}\right)$$

$$= 270x - 1800 - 12x^{\frac{3}{2}}, \quad 0 < x \le 200$$

Max when $\frac{dP}{dx} = 0$ or at end points

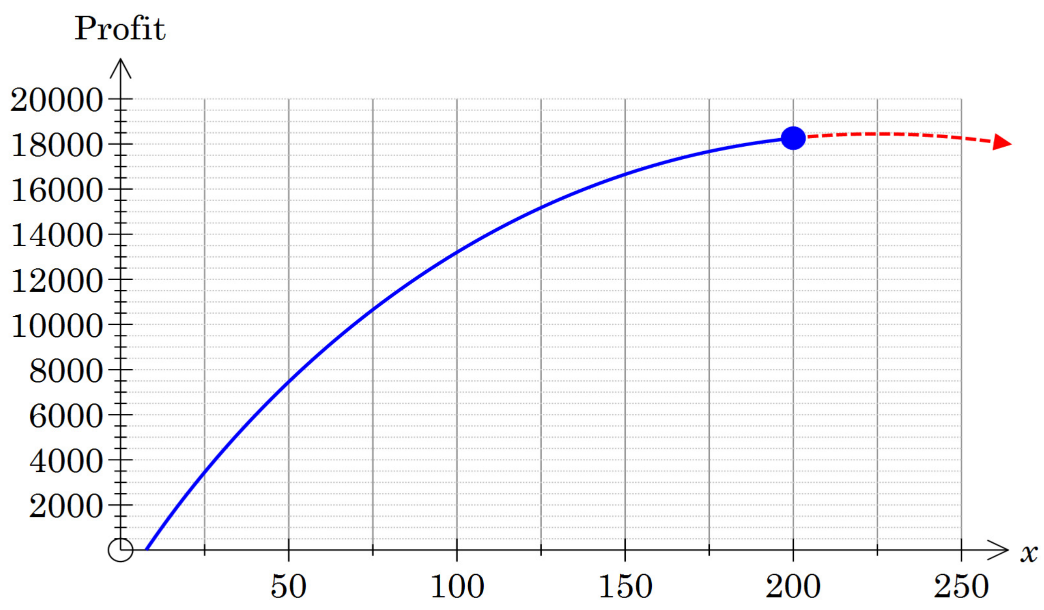
$$\frac{dP}{dx} = -18\sqrt{x} + 270$$

$$\frac{dP}{dx} = 0 \Rightarrow x = 225$$

$$\left.\frac{d^2P}{dx^2}\right|_{x=225} < 0$$

$\therefore$ Local max when x=225

Since $x \le 200$ and function is increasing the maximum profit of \$18 258.87 will be when 200 chairs are made.



Activity 12 Silos'r'us

1.

a) $\tan 35° = \dfrac{h}{r}$

$h = r\tan 35°$

b) $\cos 35° = \dfrac{r}{s}$

$s = \dfrac{r}{\cos 35°}$

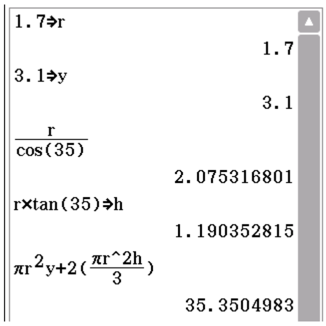
2. There is a cylinder and two cones

$$V = \pi r^2 y + 2\left(\frac{1}{3}\pi r^2 \times r\tan 35°\right)$$

$$= \pi r^2 y + \frac{2}{3}\pi r^3 \tan 35°$$

$$= \pi 1.7^2 \times 3.1 + \frac{2}{3}\pi 1.7^3 \tan 35°$$

$$= 35.4 \text{ m}^3$$

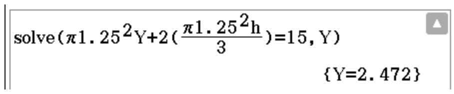


3. Note in the ClassPad solution: Y is used as y was defined earlier. Alternatively use Memory Manager to clear the value of y.

a) $15 = \pi 1.25^2 y + \dfrac{2}{3}\pi 1.25^3 \tan 35°$

$$y = \frac{15 - \frac{2}{3}\pi 1.25^3 \tan 35°}{\pi 1.25^2}$$

$$= 2.47 \text{ m}$$

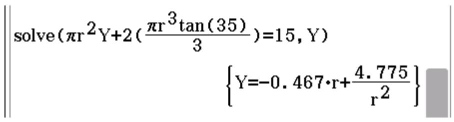


b) $V = \pi r^2 y + \dfrac{2}{3}\pi r^3 \tan 35°$

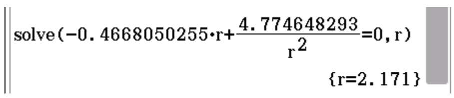
$$y = \frac{V - \frac{2}{3}\pi r^3 \tan 35°}{\pi r^2}$$

$$= \frac{15}{\pi r^2} - \frac{2r\tan 35°}{3}$$

$$= \frac{4.77}{r^2} - 0.467r$$



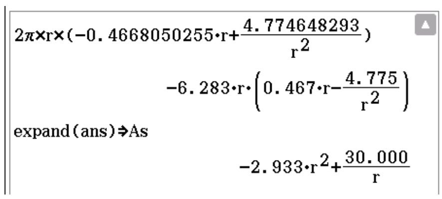
c) Max radius when height is 0.
$0 \le r \le 2.17$



d) $A = 2\pi r y$

$$= 2\pi \times 2.5 \times \left(\frac{0.467}{r^2} - 4.77r\right)$$

$$= -2.93r^2 + \frac{30}{r}$$

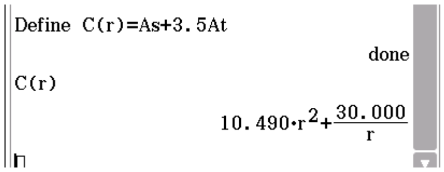


e) $A = \pi r \dfrac{r}{\cos 35°}$

$$= 3.84r^2 \text{ m}^2$$

f) $C = k\left(-2.93r^2 + \frac{30}{r} + (1.5+2)\left(3.84r^2\right)\right)$

$= k\left(10.49r^2 + \frac{30}{r}\right)$

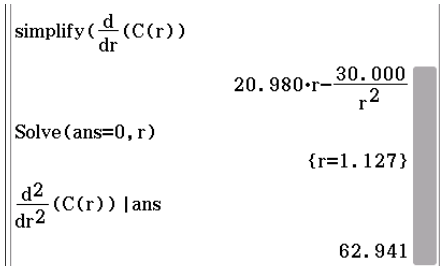


g) $\frac{dC}{dr} = -30kr^{-2} + 21.0kr$

Stationary points when $\frac{dC}{dr} = 0$

$\Rightarrow r = 1.13$

is a minimum as $\left.\frac{d^2C}{dr^2}\right|_{r=1.13} > 0$

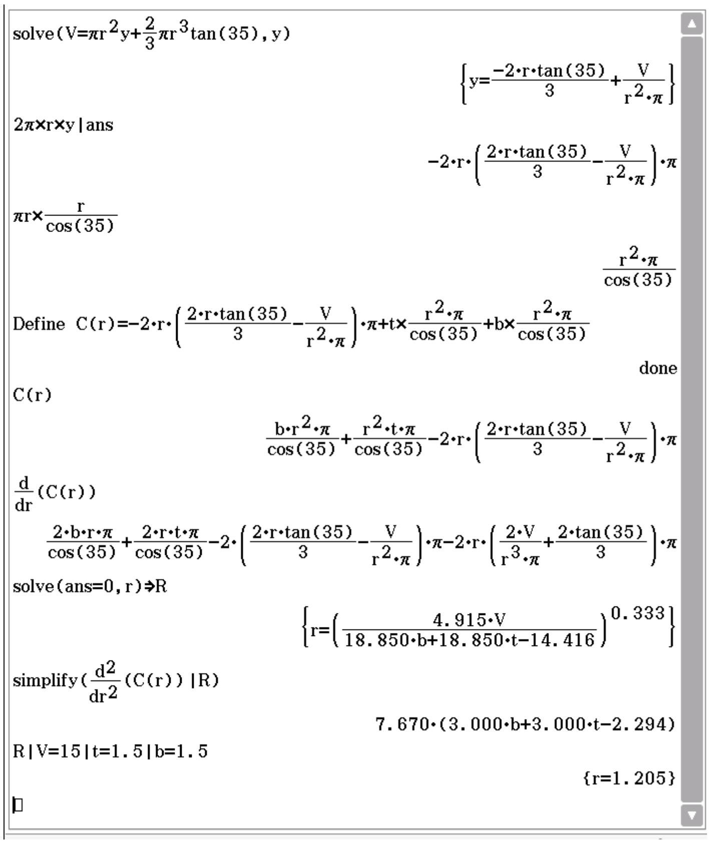


4. $V = 2\pi ry + 2\times\frac{1}{3}\pi r^2(r\tan 35°)$

$y = \frac{V}{\pi r^2} - \frac{2\tan(35°)}{3}x$

$\text{Cost} = k\left(r^2\left(\frac{\pi(b+t)}{\cos(35°)} - \frac{4\pi\tan 35°}{3}\right) + \frac{2V}{r}\right)$

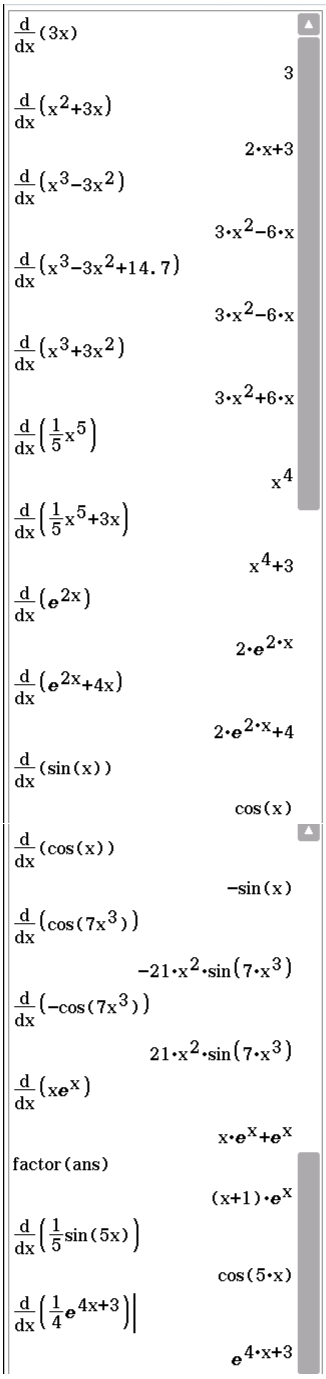
max when $r = \sqrt[3]{\frac{3V\cos 35°}{\pi(3(b+t) - 4\sin 35°)}}$



Activity 13 What might the function be?

1.

	Function	Derivative
a)	$3x$	3
b)	x^2+3x	$2x+3$
c)	x^3-3x^2	$3x^2-6x$
d)	$x^3-3x^2+14.7$	$3x^2-6x$
e)	x^3+3x^2	$3x^2+6x$
f)	$\frac{1}{5}x^5$	x^4
g)	$\frac{1}{5}x^5+3x$	x^4+3
h)	e^{2x}	$2e^{2x}$
i)	$e^{2x}+4x$	$2e^{2x}+4$
j)	$\sin x$	$\cos x$
k)	$\cos x$	$-\sin x$
l)	$\cos\left(7x^3\right)$	$-21x^2\sin\left(7x^3\right)$
m)	$-\cos\left(7x^3\right)$	$21x^2\sin\left(7x^3\right)$
n)	xe^x	xe^x+e^x
o)	xe^x	$(x+1)e^x$
p)	$\frac{1}{5}\sin 5x$	$\cos 5x$
q)	$\frac{1}{4}e^{4x+3}$	e^{4x+3}



2.

a) $\dfrac{x^4}{4}+2x+6$

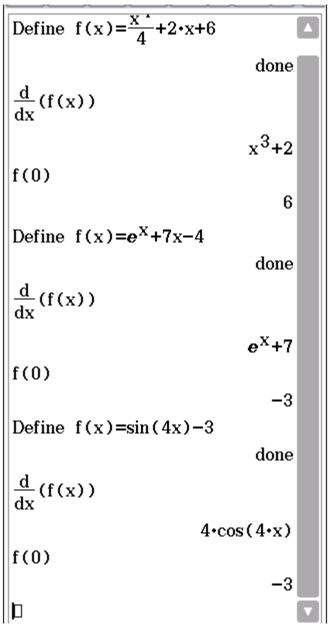
b) e^x-7x-4

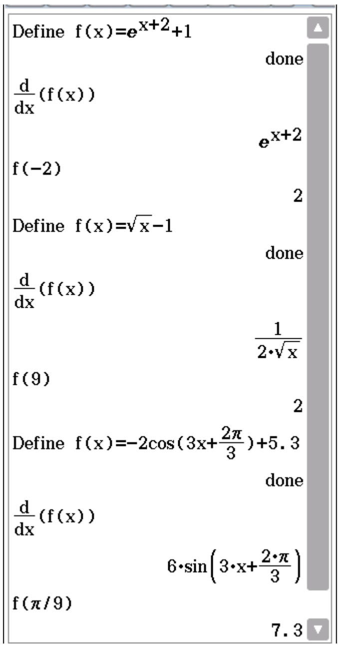
c) $\sin(4x)-3$

d) $e^{x+2}+1$

e) $\sqrt{x}-1$

f) $-2\cos\left(3x+\dfrac{2\pi}{3}\right)+5.3$





3.

a) $\dfrac{1}{n+1}x^{n+1}+c$

b) e^x+c

c) $-\cos x+c$

d) $\sin x+c$

e) $2\sin x-\cos x+c$

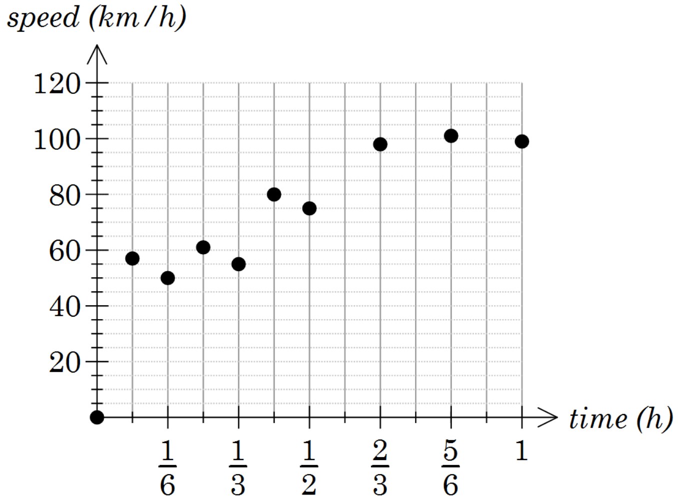
f) $\dfrac{1}{a(n+1)}(ax+b)^{n+1}+c$

g) $\dfrac{1}{a}e^{ax+b}+c$

h) $\dfrac{1}{a}\sin(ax+b)+c$

Activity 14 Are we there yet?

1. a)



b) 75-80 km/h would be reasonable estimates

c) 80 km

d) Travelling at about 60 km/hr for the first 20 minutes, about 80 km/h for the next 20 minutes and 100 for the last 20 minutes. I.e. there are no stops for traffic lights or other reasons.

2. a) (i) $50/6 \approx 8.3$ km

(ii) $\dfrac{50}{6}$ km

b) They are both 50 km multiplied by $\dfrac{1}{6}$ of an hour

c) (i) 65 km/h

(ii) $\dfrac{50+80}{2}\times\dfrac{1}{12} = 65\times\dfrac{1}{12} \approx 5.4$ km

d) (i) $\dfrac{1}{12}$ hour (ii)5 km/h (iii) $\dfrac{5}{12}$ km

e) 53.75 km

3. a)

Time	v_{start}	v_{end}	v av	Distance)
0 – 10	0	55.6	27.8	4.6
10 – 20	55.6	88.9	72.2	12
20 – 30	88.9	100	94.4	15.7
30 – 40	100	88.9	94.4	15.7
40 – 50	88.9	55.6	72.2	12
50 – 60	56	0	27.8	4.6

b) 64.8 km

c) 66 km

d) It is increasing

e) More as there is area above the trapezia

f) (i) 56.6 km

(ii) 63.5 km

	A	B	C	D	E
1	t	v0	v1	dist	total
2	0	0	36	1.8	1.8
3	0.1	36	64	5	6.8
4	0.2	64	84	7.4	14.2
5	0.3	84	96	9	23.2
6	0.4	96	100	9.8	33
7	0.5	100	96	9.8	42.8
8	0.6	96	84	9	51.8
9	0.7	84	64	7.4	59.2
10	0.8	64	36	5	64.2
11	0.9	36	0	1.8	66
12	1				

4. a) Answers will vary

b) As n increases the total distance increases. It gets closer to $66\frac{2}{3}$ km

c) 66.7 km

d) (i) 10.4 km

(ii) 22.9 km

(iii) 33.3 km

(iv) 45.8 km

5. a) $\frac{d}{dt}\left(100t-\frac{400}{3}\left(t-\frac{1}{2}\right)^3\right)=100-400\left(t-\frac{1}{2}\right)^2$

b)

t	0	0.25	0.5	0.75	1
$s(t)$	16.7	27.1	50	72.9	83.3

c) $s(0.25)-s(0)=10.4$ distance travelled $0 \le t \le 0.25$

$s(0.5)-s(0.25)=22.9$ distance travelled $0.25 \le t \le 0.5$

$s(0.5)-s(0)=33.3$ distance travelled $0 \le t \le 0.5$

$s(0.75)-s(0.25)=45.8$ distance travelled $0.25 \le t \le 0.75$

i.e. the difference between the antiderivative calculated at the end points of the interval.

6. a) answers will vary slightly with the number of intervals used

Velocity function	In first hour	In second hour	In first two hours
$v(t)=100(1-e^{-2t})$	56.7	94.1	150.8
$v(t)=100\sin\left(\frac{\pi t}{2}\right)$	63.6	63.6	127.2

b) & c)

Velocity function	Anti-derivative	$t=0$	$t=1$	$t=2$
$v(t)=100(1-e^{-2t})$	$100t+50e^{-2t}$	50	106.8	200.9
$v(t)=100\sin\left(\frac{\pi t}{2}\right)$	$v(t)=-\frac{200}{\pi}\cos\left(\frac{\pi t}{2}\right)$	–63.7	0	63.7

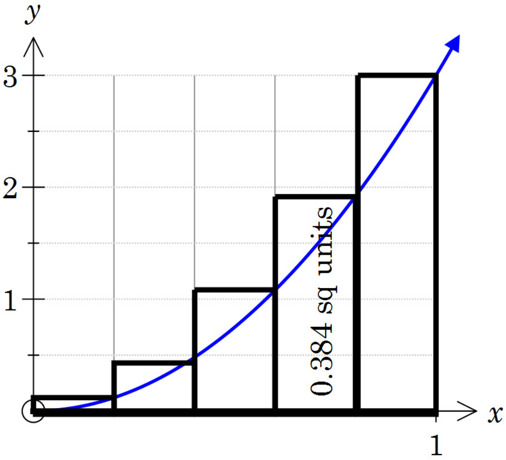
Note: other anti-derivatives are possible by adding a constant.

d) The difference between the anti-derivative calculated at the end points of the interval is approximately equal to the distance travelled.

Activity 15 The fundamental theorem of calculus

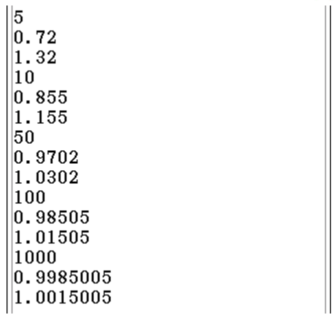
1. a) The sum of the areas of the four rectangles is 0.72

   b) See graph



   c)

5	0.72	1.32
10	0.855	1.155
50	0.9702	1.0302
100	0.98505	1.01505
1000	0.9985005	1.0015005



   d) The values get closer together.

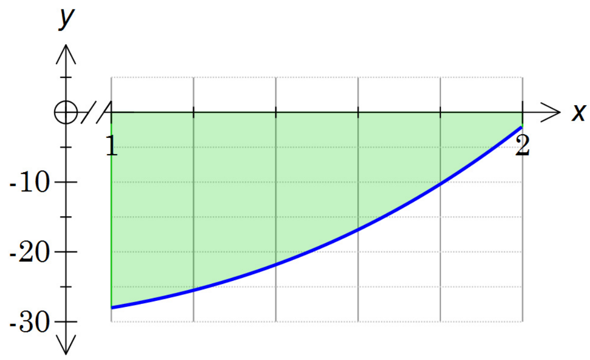
   e) 1 sq. unit.

2. a)

Interval	Area
$0 \le x \le 2$	8
$0 \le x \le 5$	125
$2 \le x \le 5$	117
$3 \le x \le 10$	973
$-1 \le x \le 1$	2

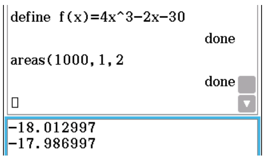
   b) Substitute the domain extremities and calculate the difference.

3. a) See graph



   b) As the y-values are negative the values are opposite and so the area lies between 15.28 and 20.48

   c) Approximately 18 units^2

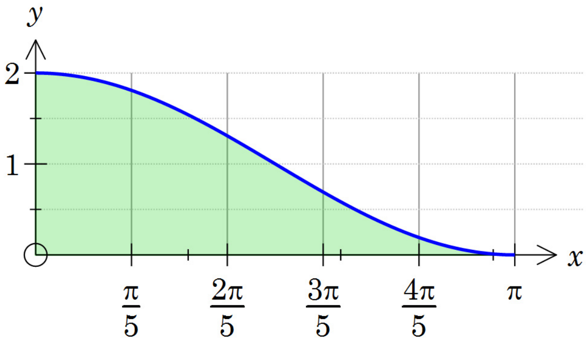


   d) $A(x) = x^4 - x^2 - 30x$, $A(2) - A(1) = 18$

e)

Interval	Area (units^2)
$0 \le x \le 1$	30
$0 \le x \le 2$	48
$0.5 \le x \le 1.5$	27

4. a) See graph

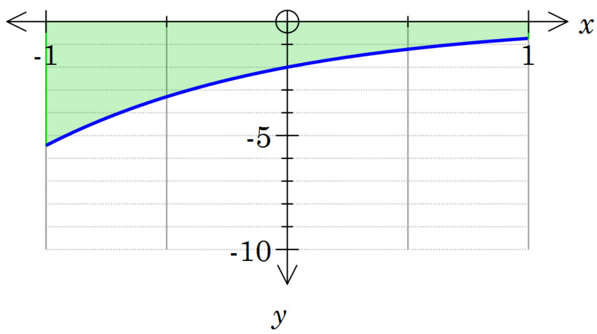


b) As the graph is decreasing the "lower area" is a larger value than the "upper area"

c) 3.1 units^2

d) $A(x) = x + \sin(x)$
$A(\pi) - A(0) = \pi$

5.



a)

b) The function is negative and increasing. The values are negative because the function is negative, and the absolute value of the lower area is higher as the rectangles have a greater height (because the function is increasing).

c) 4.7 units^2

d) $A(x) = 2e^{-x}$
$A(1) - A(-1) = 2e^{-1} - 2e^{1} = -4.7$

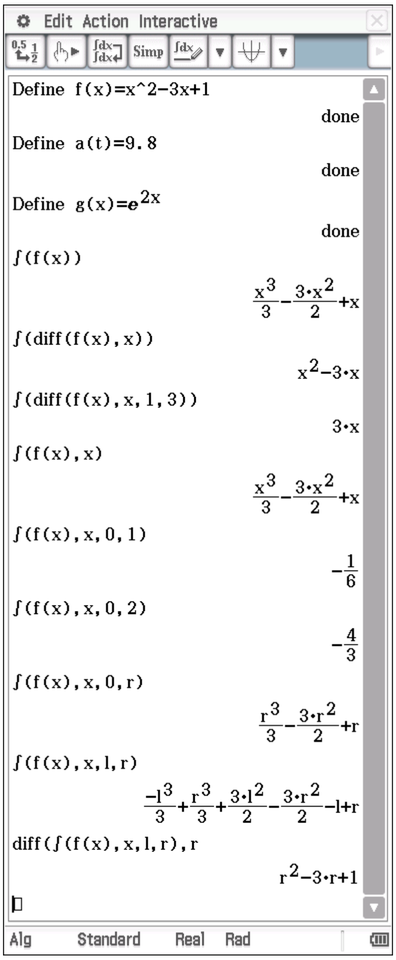
6.

$f(x) \ge 0$	$f'(x) \ge 0$	"upper" rectangles – "lower" rectangles
$f(x) \le 0$	$f'(x) \ge 0$	–("lower" rectangles – "upper" rectangles)
$f(x) \ge 0$	$f'(x) \le 0$	"lower" rectangles – "upper" rectangles
$f(x) \le 0$	$f'(x) \le 0$	–("upper" rectangles – "lower" rectangles)

Activity 16 Integrate

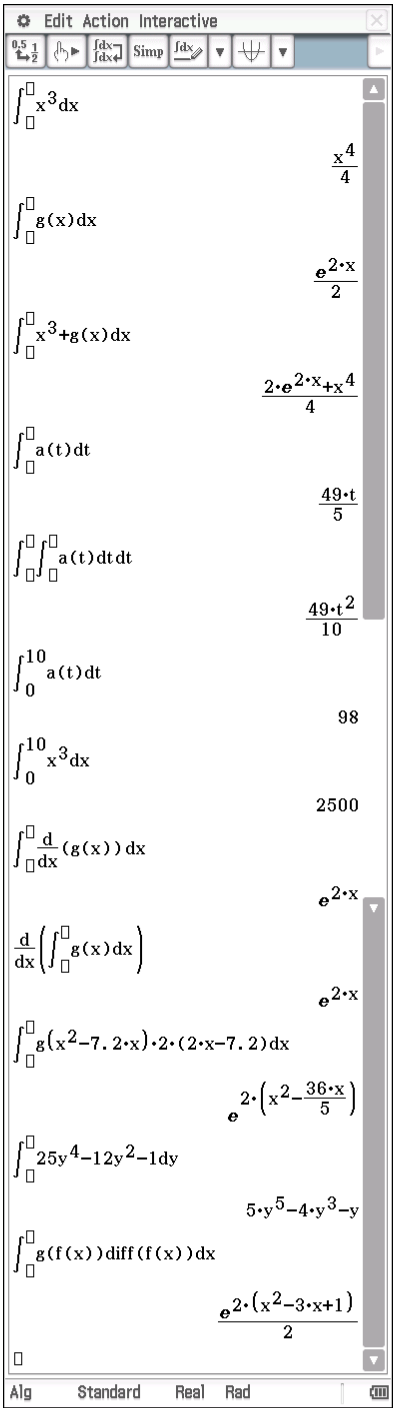
1.

a) Explanations will vary depending upon prior knowledge



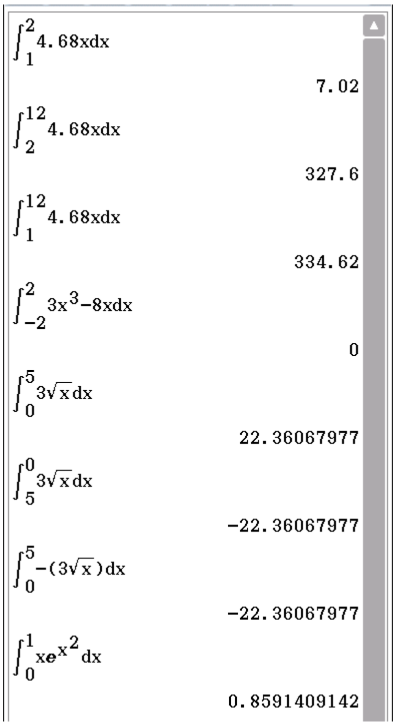
b) The ∫ command calculates the anti-derivative or integral of a function, where possible.
The first parameter is the function. The second is the variable, if omitted it is assumed to be *x*.
The next two parameters enable calculation of a definite integral with a lower or left boundary and an upper or right boundary.

2.



Note that all indefinite integrals should have +c attached!

3.



4. $$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

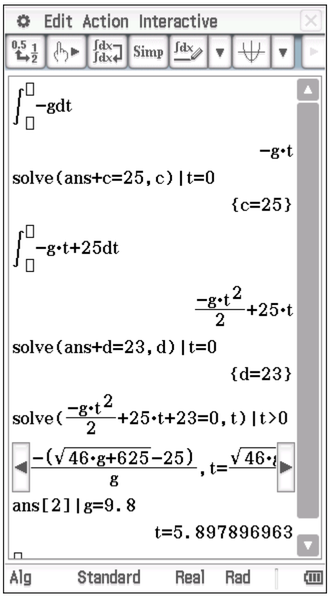
$$\int_a^b -f(x)dx = -\int_a^b f(x)dx$$

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

Activity 17 Distance from acceleration

1. a)

Acceleration	$-g$
Height at time $t=0$	23
Velocity at $t=0$	25



b) $v = 25 - gt$

c) $s = -\frac{1}{2}gt^2 + 25t + 23$

d)

Solve $25t + 23 - \frac{1}{2}gt^2 = 0$

$$t = \frac{-23 \pm \sqrt{25^2 - 4\left(-\frac{1}{2}\right)23}}{-g}$$

$$= \frac{23 + \sqrt{25^2 + 46}}{g}, t > 0$$

$= 5.9s$ (2 s.f.)

e) $s = ut + \frac{1}{2}at^2 + s_0$

2.

a) $v(0) = -20\text{ cms}^{-1}$, $s(0) = 100\text{ cm}$

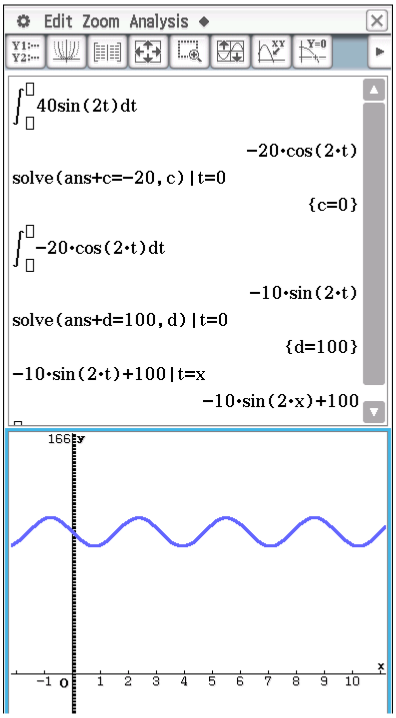
b) $v = -20\cos 2t$

c) 20 cms^{-1}

d) $s = -10\sin 2t + d$
$s(0) = 100 = -10\sin 0 + d$
$d = 100$
$s = -10\sin 2t + 100$

e) 0 cms^{-1}

f) Periodic, with mean height 100 cm and period π seconds



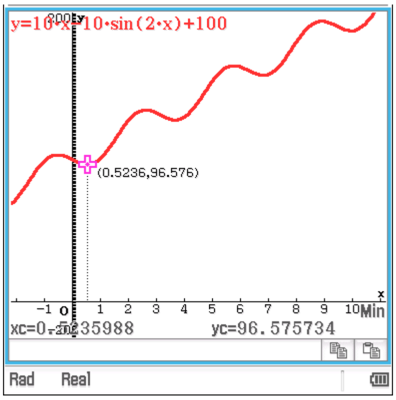
g)

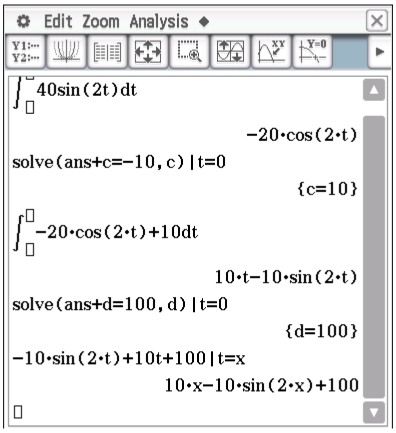
(i) $s = 10t - 10\sin 2t + d$

$s(0) = 100 = 10 \times 0 - 10\sin 0 + d$

$d = 100$

$s = 10t - 10\sin 2t + 100$





(ii) The motion is periodic while advancing to the right.

3.

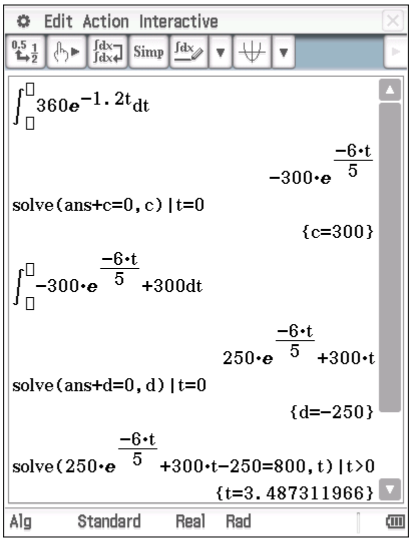
a) $v = 300 - 300e^{-1.2t}$

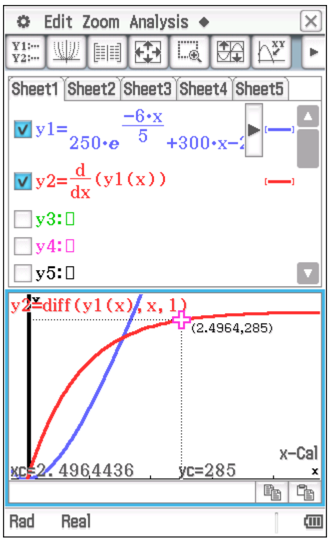
$s = 250e^{-1.2t} + 300t - 250$

b) 300 cm/s

c) 2.5 s

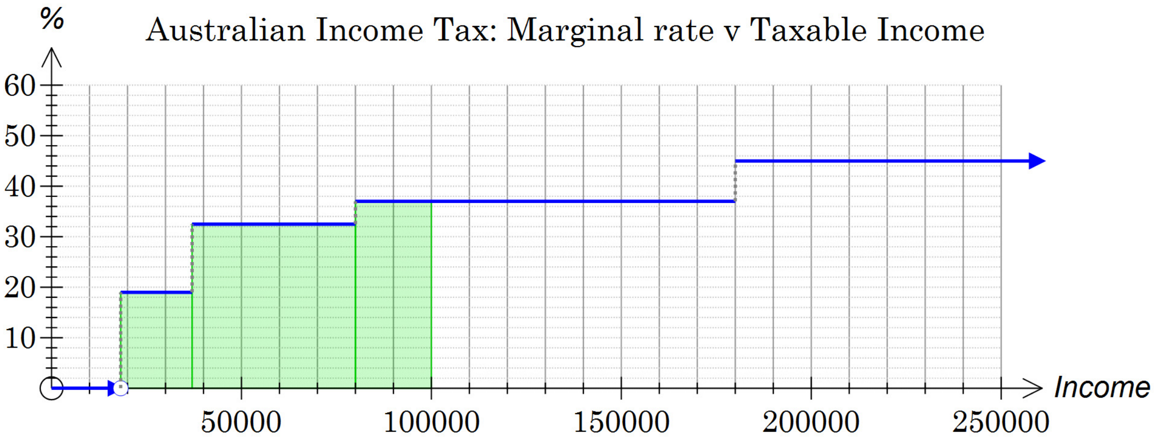
d) 3.49 s





Activity 18 Tax time

1. a)



c) (i) $342
(ii) $4547
(iii) $24947
(iv) $54547

2.

a)
(i) $8.91
(ii) $1307.20
(iii) $18 405
(iv) $54 405

b) No-one pays more tax.

c) $$\text{TaxRate}(\$x) = \begin{cases} 0, & x \le 15000 \\ \dfrac{x - 15000}{65000} \times 0.45, & 15000 < x \le 80000 \\ 0.45, & x > 80000 \end{cases}$$

(i) $86.54
(ii) $2163.46
(iii) $23 625
(iv) $59 625
The $180 000 income would pay $5078 more.

3.

a)
(i) $12.81
(ii) $1474.24
(iii) $24 750
(iv) $60 750

b) $180 000 pays more tax.

4. a)

$$T(x)=\int 0.325dx = 0.325x + c$$

$$T(37000)=0.325(37000)+c=3572$$

$$c=-8453$$

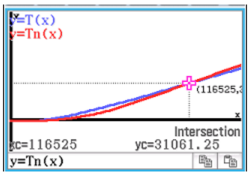
$$T(x)=0.325x-8543$$

Alternatively

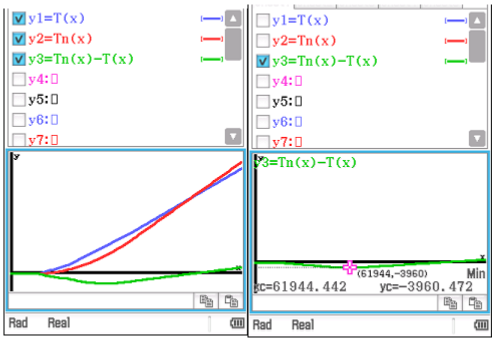
$$3572+0.325(x-37000)=0.325x-8543$$

b) $$Tn(x)=\begin{cases} 0, & x \le 15000 \\ \dfrac{9x^2}{2600000}-\dfrac{27x}{260}+778.85, & 15000 < x \le 80000 \\ 0.45x-21375, & x > 80000 \end{cases}$$

c) Incomes between \$15 000 and \$18 412 and above \$116 525 will pay more tax.



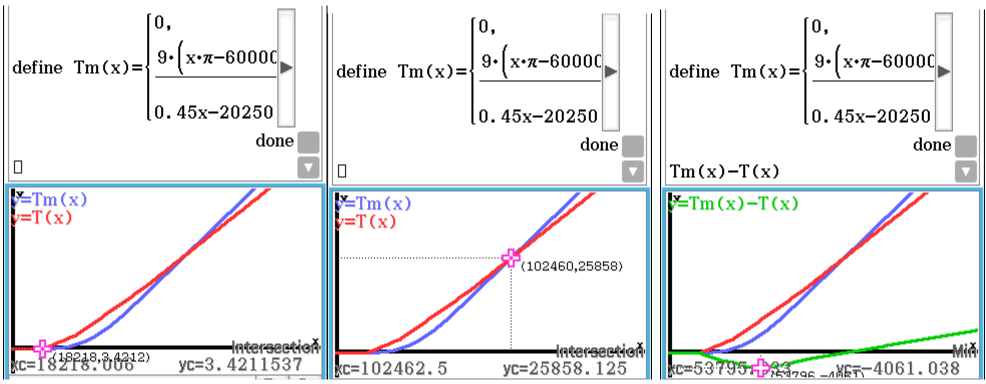
d) A person on \$61944 will pay \$3960 less tax.



5.

a) $$Tm(\$x)=\begin{cases} 0, & x \le 15000 \\ 0.225\left(x-\dfrac{60000}{\pi}\cos\left(\dfrac{\pi(x-45000)}{60000}\right)\right)-3375, & 15000 < x \le 75000 \\ 0.45x-20250, & x > 75000 \end{cases}$$

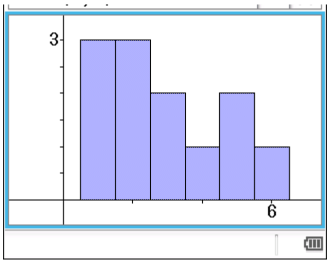
b) Between \$15 000 and \$18 218 and above \$102 460 pay more tax

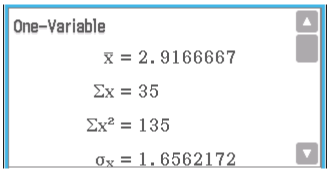


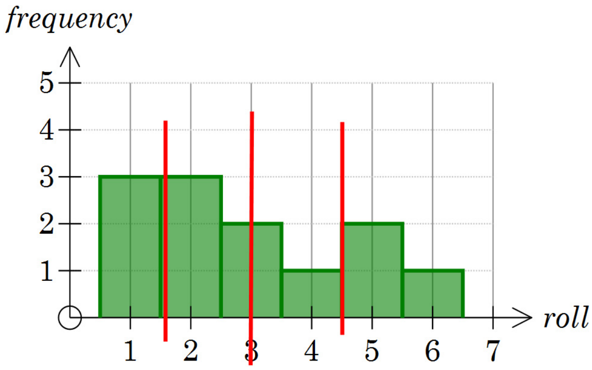
c) Biggest gain is \$53 796 income which will pay \$4061 less tax.

Activity 19 Rolling Dice

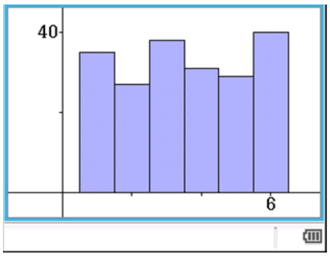
1. Results will vary. For example:

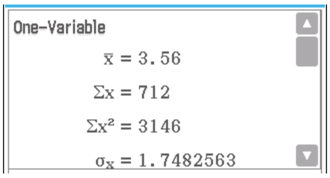


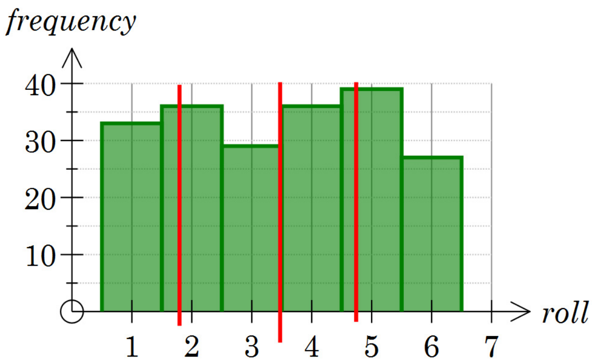




2. Results will vary. For example:

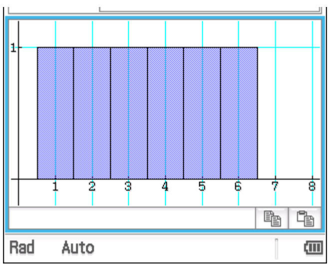


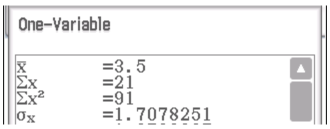


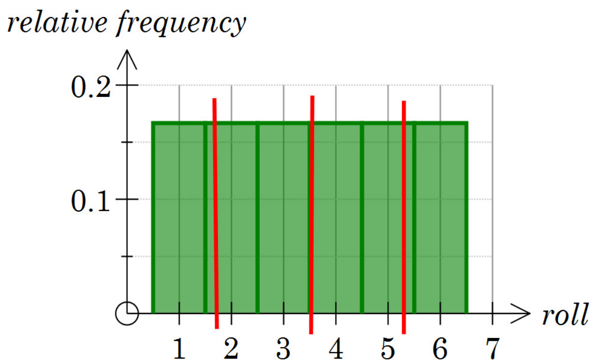


c) The distribution is likely to look more uniform as in the examples above.

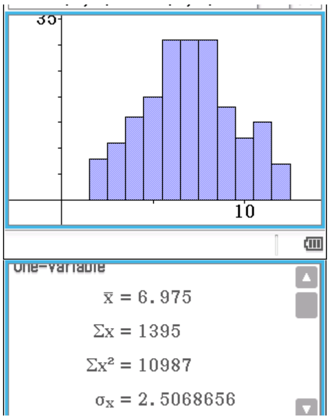
3.

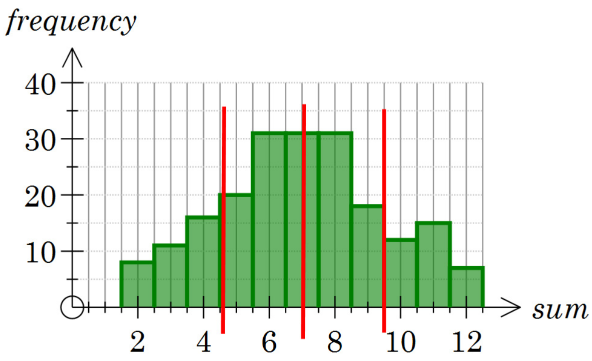






4. Results will vary. For example:





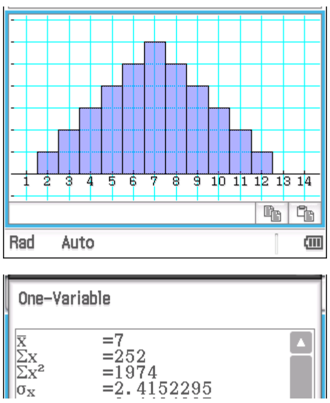
5.

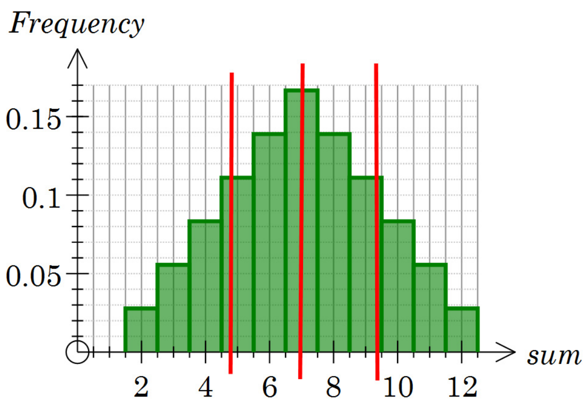
a)

		Die 1					
		1	2	3	4	5	6
Die 2	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

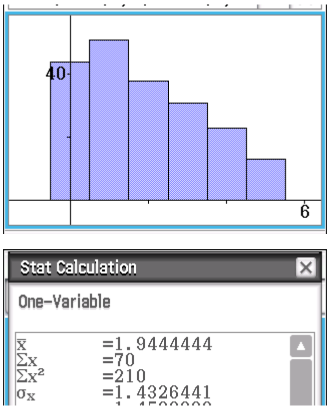
Sum	Frequency
2	1
3	2
4	3
5	4
6	5
7	6
8	5
9	4
10	3
11	2
12	1

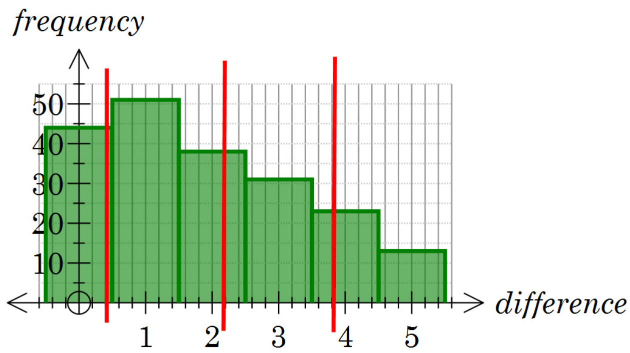
b) & c)





6. a)



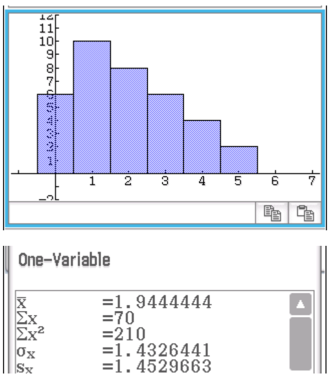


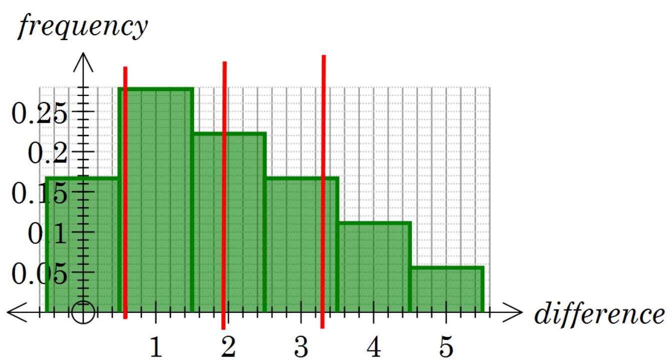
c)

		Die 1					
		1	2	3	4	5	6
Die 2	1	0	1	2	3	4	5
	2	1	0	1	2	3	4
	3	2	1	0	1	2	3
	4	3	2	1	0	1	2
	5	4	3	2	1	0	1
	6	5	4	3	2	1	0

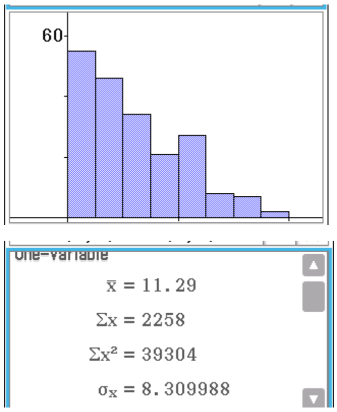
Difference	Frequency
0	6
1	10
2	8
3	6
4	4
5	2

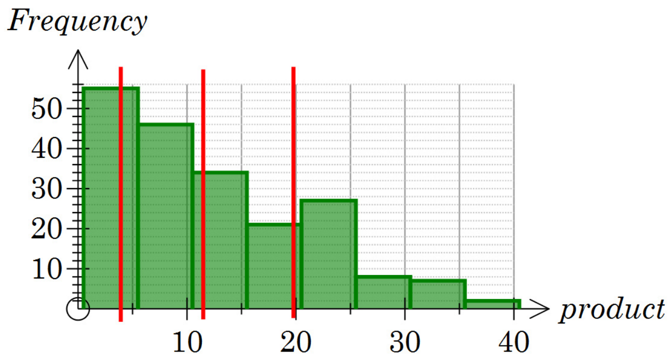
d)





7. a)



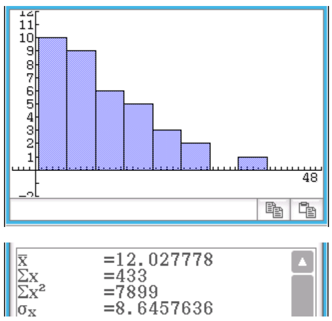


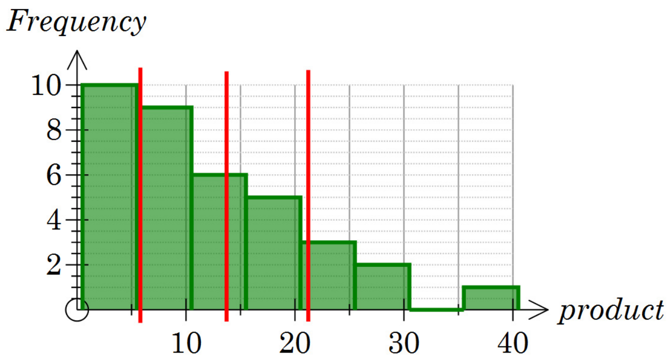
c)

		Die 1					
		1	2	3	4	5	6
Die 2	1	1	2	3	4	5	6
	2	2	4	6	8	10	12
	3	3	6	9	12	15	18
	4	4	8	12	16	20	24
	5	5	10	15	20	25	30
	6	6	12	18	24	30	36

X=product	Frequency
1 – 5	10
6 – 10	9
11 – 15	6
16 – 20	5
21 – 25	3
26 – 30	2
31 – 35	0
36 – 40	1

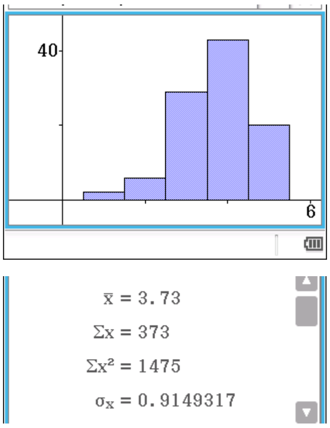
d)

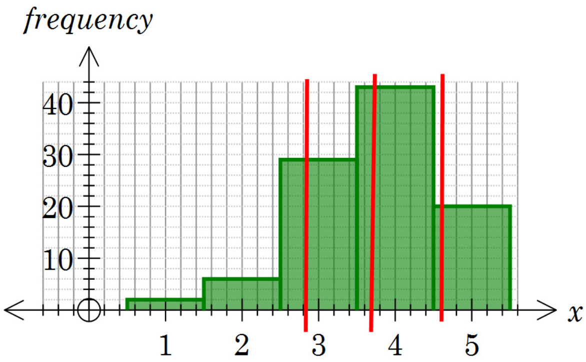




Activity 20 Up or down, the Bernoulli distribution

1. Results will vary. For example:





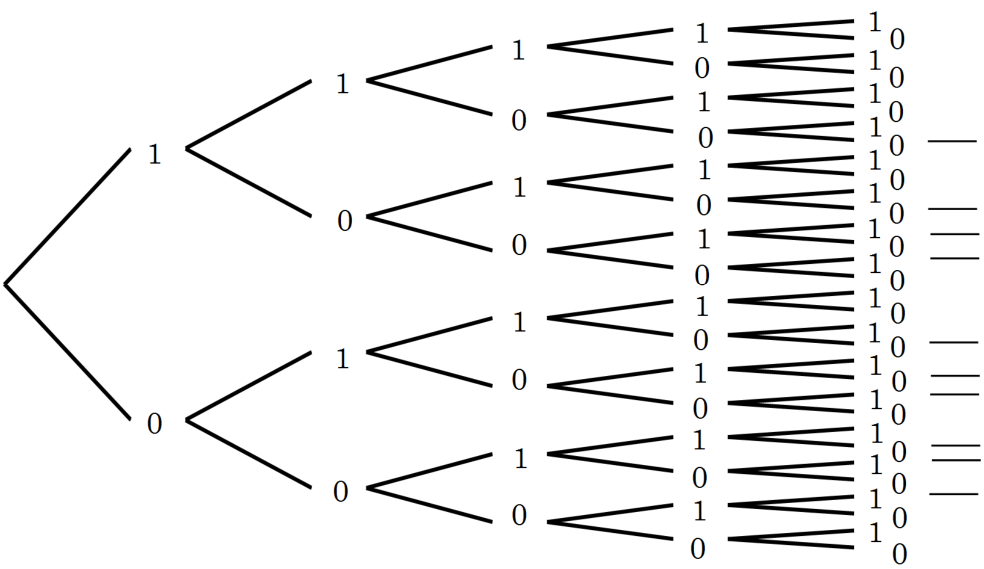
c) The histogram has a peak at 4 and is negatively skewed.

2. a) $\left(\frac{1}{4}\right)^5 = \frac{1}{1024}$

b)

face up cards	Arrangements
0	{0,0,0,0,0}
1	{1,0,0,0,0} {0,1,0,0,0} {0,0,1,0,0} {0,0,0,1,0} {0,0,0,0,1}
2	{1,1,0,0,0} {1,0,1,0,0} {1,0,0,1,0} {1,0,0,0,1} {0,1,1,0,0} {0,1,0,1,0} {0,1,0,0,1} {0,0,1,1,0} {0,0,1,0,1} {0,0,0,1,1}
3	{1,1,1,0,0} {1,1,0,1,0} {1,1,0,0,1} {1,0,1,1,0} {1,0,1,0,1} {0,1,1,0,1} {0,1,0,1,1} } {0,1,1,1,0} {1,0,0,1,1} {0,0,1,1,1}
4	{1,1,1,1,0} {1,1,1,0,1} {1,1,0,1,1} {1,0,1,1,1} {0,1,1,1,1}
5	{1,1,1,1,1}

c) & d)



e)

x	# of arrangements	P(a branch)	$P(X=x)$
0	1	$\left(\frac{1}{4}\right)^5$	$\frac{1}{4^5}=0.000977$
1	5	$\left(\frac{1}{4}\right)^4\times\left(\frac{3}{4}\right)$	$\frac{15}{4^5}=0.0146$
2	10	$\left(\frac{1}{4}\right)^3\times\left(\frac{3}{4}\right)^2$	$\frac{90}{4^5}=0.0879$
3	10	$\left(\frac{1}{4}\right)^2\times\left(\frac{3}{4}\right)^3$	$\frac{270}{4^5}=0.264$
4	5	$\left(\frac{1}{4}\right)\times\left(\frac{3}{4}\right)^4$	$\frac{405}{4^5}=0.396$
5	1	$\left(\frac{3}{4}\right)^5$	$\frac{243}{4^5}=0.237$

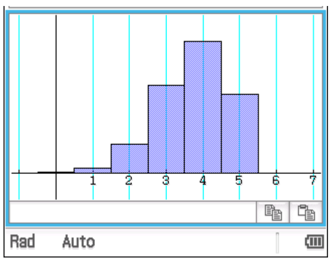
3.

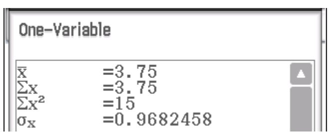
a)

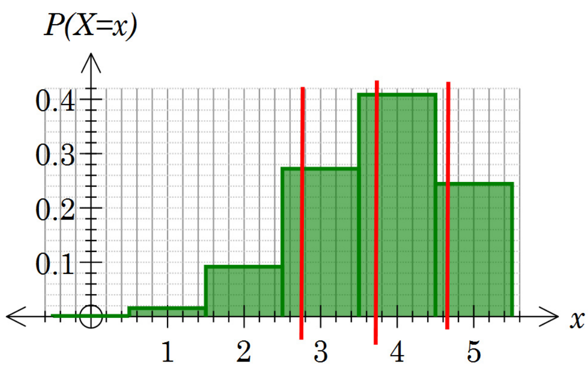
x	0	1	2	3	4	5
binomPDf(x,5,0.75)	0.000977	0.0146	0.0879	0.264	0.396	0.237

b) Identical

c) & d)







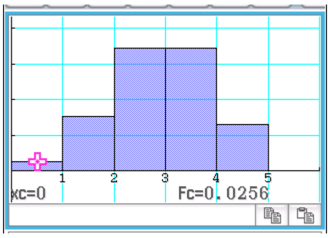
Activity 21 Bernoulli trials

1.

Number of pipes	0	1	2	3	4
Arrangements	XXXX	XXXO XXOX XOXX OXXX	XXOO XOXO XOOX OXOX OXXO OOXX	XOOO OXOO OOXO OOOX	OOOO
Probability	$\frac{1}{16}$	$\frac{4}{16}=\frac{1}{4}$	$\frac{6}{16}=\frac{3}{8}$	$\frac{4}{16}=\frac{1}{4}$	$\frac{1}{16}$

2. b)

(i)



(ii)

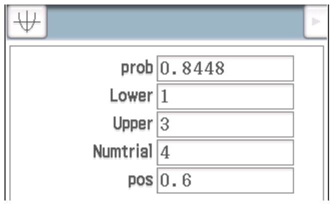
# of pipes	Probability
0	0.0256
1	0.1536
2	0.3456
3	0.3456
4	0.1296

c)

(i) 0.8208

(ii) 0.8704

(iii) 0.8448



3. binomPD gives the probability of a particular number of successes whereas binomCD gives the probability for a range of values.

4.

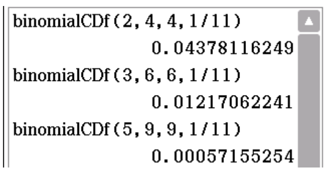
a) 11 arrangements have 2 or more pipes with water. One of these is correct.

b)

(i) 0.044

(ii) 0.012

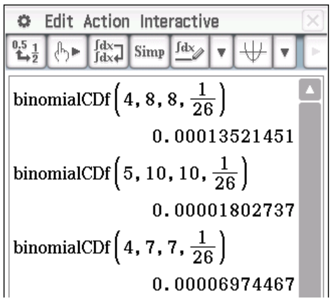
(iii) 0.00057



5.

a) There are $2^5 = 32$ arrangements of the five pipes to carry water. Of these, $\binom{5}{0} + \binom{5}{1} = 6$ have fewer than two pipes carrying water. Hence 26 different possible arrangements remain, so Max has a $\frac{1}{26} \approx 0.0384$ chance of a correct guess on a single trial.

b) P(at least 4 out of 8) ≈ 0.00014



c) If the reasoning is there is an even number of trials then 10 trials are required and
P(at least 5) = 1.8×10^{-5}
However P(at least 4 out of 7) is 7.0×10^{-5} so the least number of trials is 7.

Activity 22 What is log?

1. a)

Number (x)	x as a power of 10	log x
10	10^1	1
10000	10^4	4
1	10^0	0
0.01	10^{-2}	-2
0.001	10^{-3}	-3
$\frac{1}{10}$	10^{-1}	-1
$\sqrt{10}$	$10^{\frac{1}{2}}$	$\frac{1}{2}$
$\left(\sqrt{10}\right)^3$	$10^{\frac{3}{2}}$	$\frac{3}{2}$

b) The log is the exponent when the number is written as a power of 10.

2.

a)

Number (x)	Base (b)	x written as a power of the base	$\log_b x$
9	3	3^2	2
81	3	3^4	4
1024	2	2^{10}	10
28561	13	13^4	4
125	5	5^3	3
$\sqrt{5}$	5	$5^{\frac{1}{2}}$	$\frac{1}{2}$
16	2	2^4	4
8	4	$4^{1.5}$	1.5
5.0649…	3.7	$3.7^{1.24}$	1.24

b) The log to the base b is the exponent when the number is written as a power of b.

c)

(i) a

(ii) x

d) x

Activity 23 Log laws

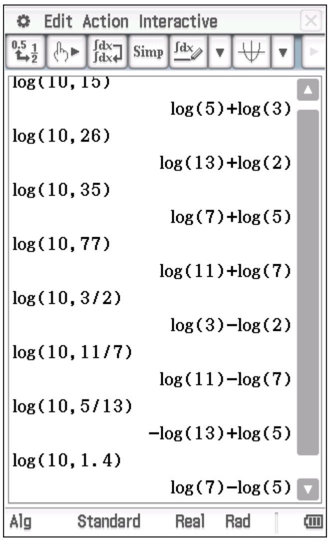
1. a) i) log5 + log3

   (ii) log13 + log2

   (iii) log7 + log5

   (iv) log11 + log7

   b) $\log(a \times b) = \log a + \log b$



2. a) i) log3 – log2

   (ii) log11 – log7

   (iii) log5 – log13

   (iv) log7 – log5

   b) $\log\left(\dfrac{a}{b}\right) = \log a - \log b$

3. a) i) 2log7

   (ii) 3log5

   (iii) –2 log3

   (iv) 6log7

   (v) $a$log3

   b) $\log(a^x) = x \log a$

4. a) log3 + log5 + log7

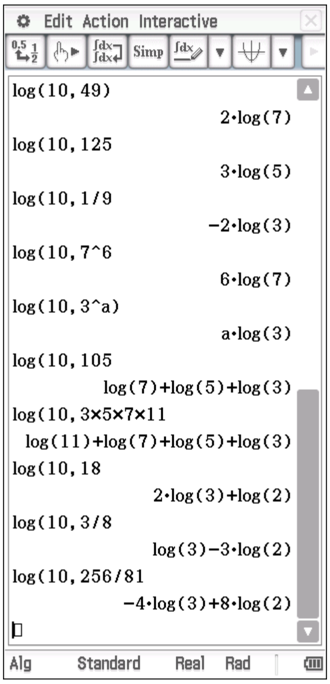
   b) log2 + log7 + log11

   c) log3 + log5 + log7 + log11

   d) log2 + 2 log3

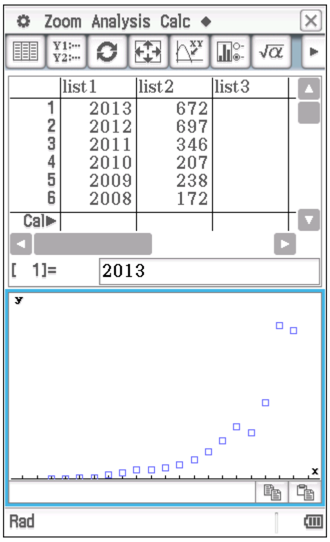
   e) log3 – 3log2

   f) 8log2 – 4log3

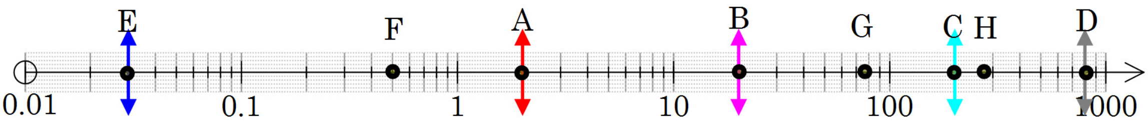


Activity 24 Growth of the WWW

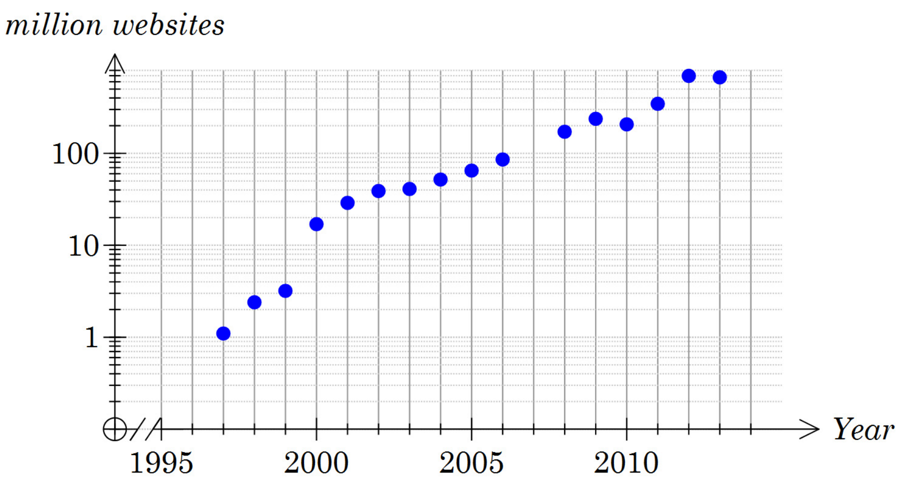
1.



2. a) A(2,0), B(20,0), C(200,0) D(800,0)



3.

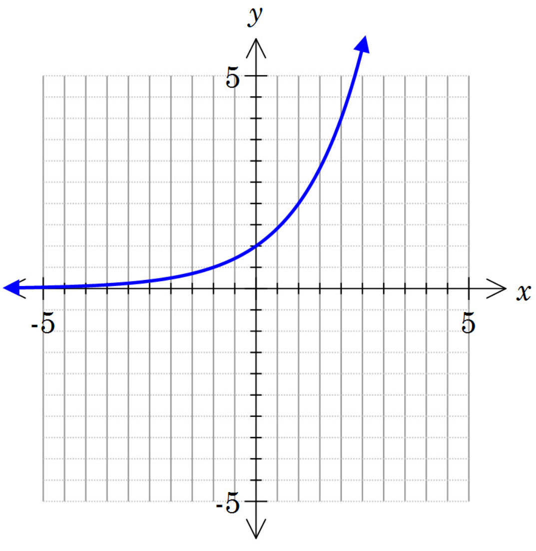


4. So we can see trends in the early years.

In the first graph the large values dominate and we can see little of the trends with the small number of websites. Using a log scale for the number of websites lets us see the growth for the early years as well. Notice that exponential growth appears linear on a semi-logarithmic graph.

Activity 25 Key features of logarithmic functions

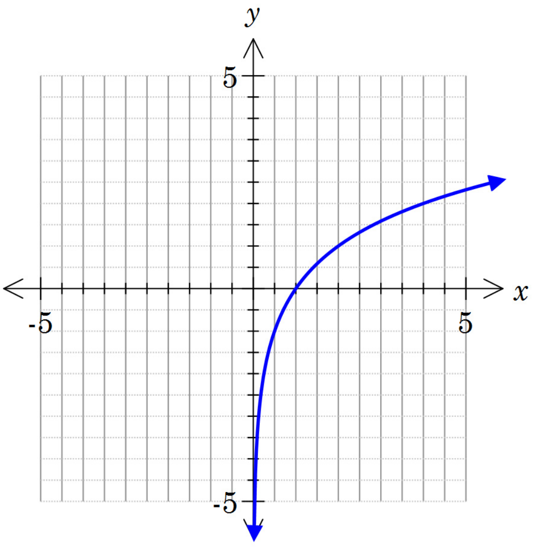
1. $y = 2^x$



Asymptote: $y = 0$

Intercept(s): (0,1)

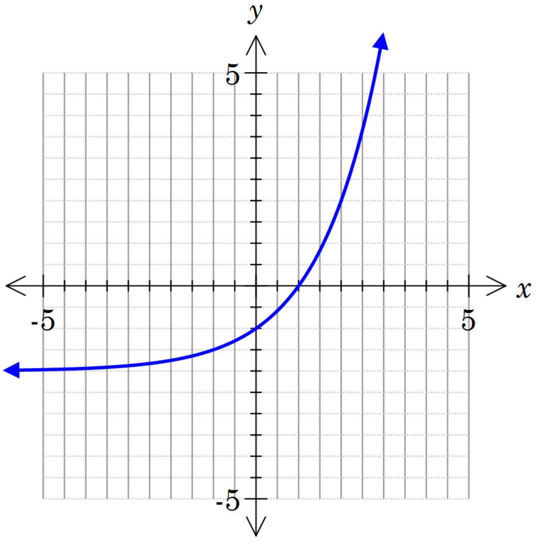
2. $y = \log_2 x$



Asymptote: $x = 0$

Intercept(s): (1,0)

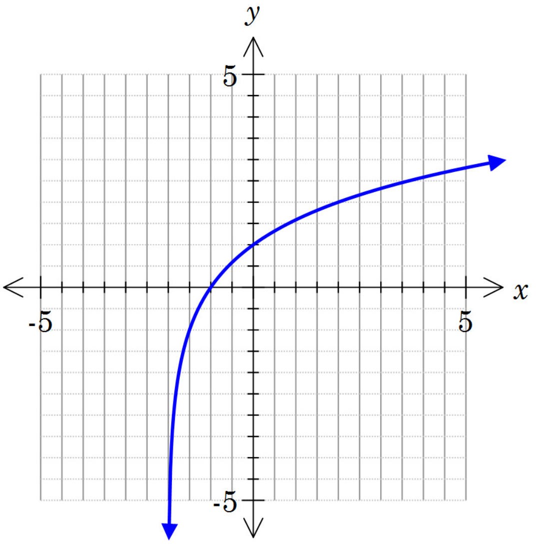
3. $y = 2^x - 2$



Asymptote: $y = -2$

Intercept(s): (0,–1)

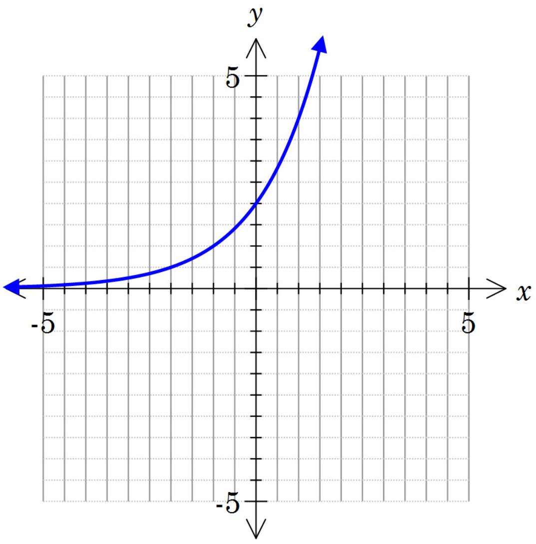
4. $y = \log_2(x+2)$



Asymptote: $x = -2$

Intercept(s): (–1,0)

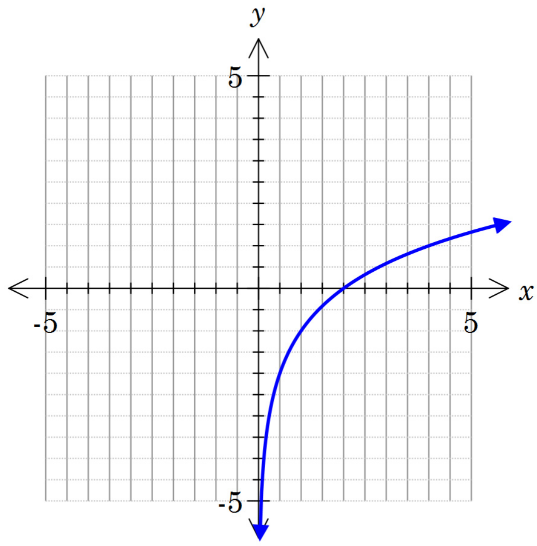
5. $y = 2^{x+1}$



Asymptote: $y = 0$

Intercept(s): (0,2)

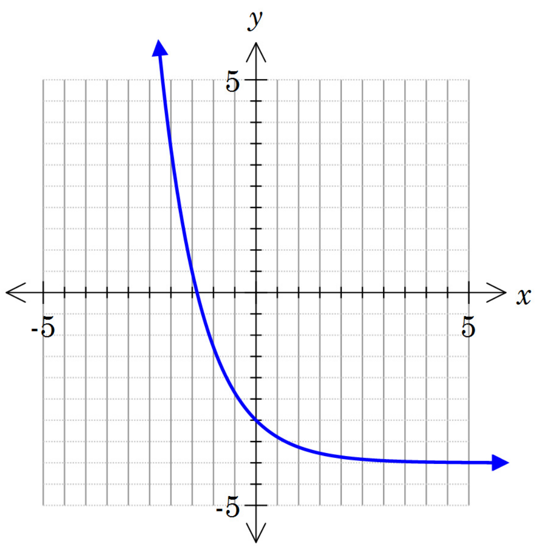
6. $y = \log_2 x - 1$



Asymptote: $x = 0$

Intercept(s): (2,0)

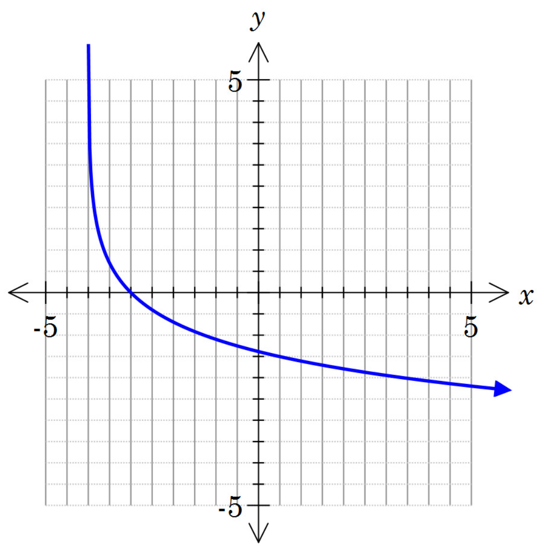
7. $y = e^{-x} - 4$



Asymptote: $y = -4$

Intercept(s): $(0,-3)$ $(-\ln(4),0)$

8. $y = -\log_e(x + 4)$

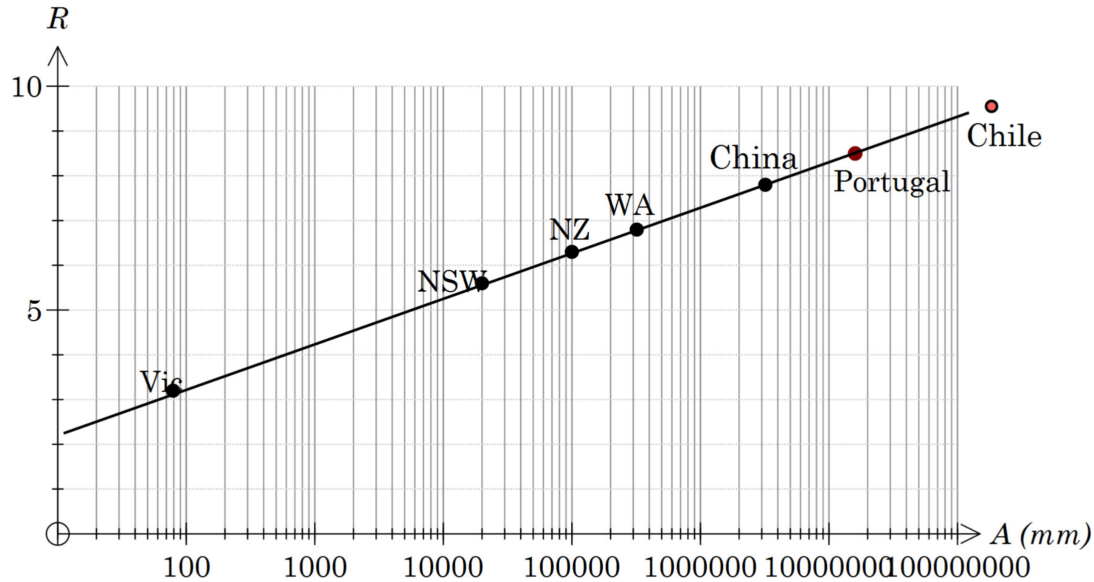


Asymptote: $x = -4$

Intercept(s): $(-3,0)$ $(0,-\ln(4))$

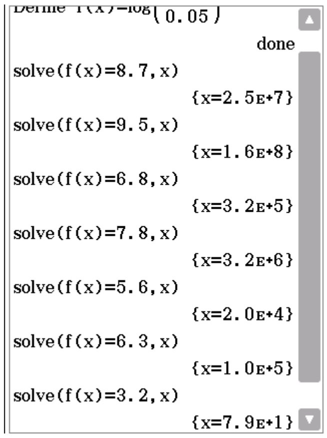
Activity 26 Applications of logs

1. a)



b)

Earthquake	R	A
Lisbon, Portugal	8.5 - 9	1.6×10^{7} to 5×10^{7}
Valdivia, Chile	9.5	1.6×10^{8}
Meckering WA	6.8	3.2×10^{5}
Tangshan , China	7.8	3.2×10^{6}
Newcastle NSW	5.6	2.0×10^{4}
Christchurch NZ	6.3	1.0×10^{5}
Melbourne, Vic	3.2	79



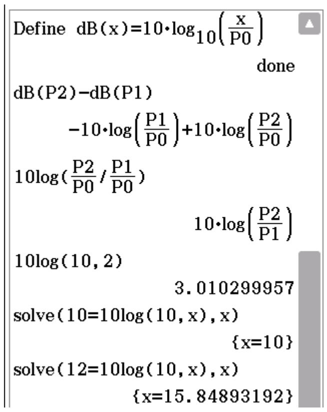
2. a) $dB_2 - dB_1 = 10\log\left(\frac{P_2}{P_0}\right) - 10\log\left(\frac{P_1}{P_0}\right)$

$= 10\log\left(\frac{P_2}{P_0} \div \frac{P_1}{P_0}\right) = 10\log\left(\frac{P_2}{P_1}\right)$

b) ~3dB (or $10\log 2$)

c) 10 times the intensity

d) ~16 times the intensity ($10^{1.2}$)

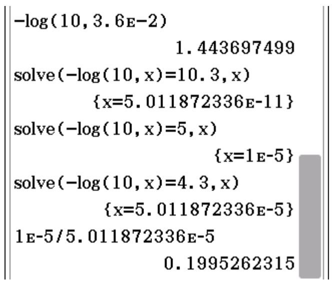


3.

a) ~1.4

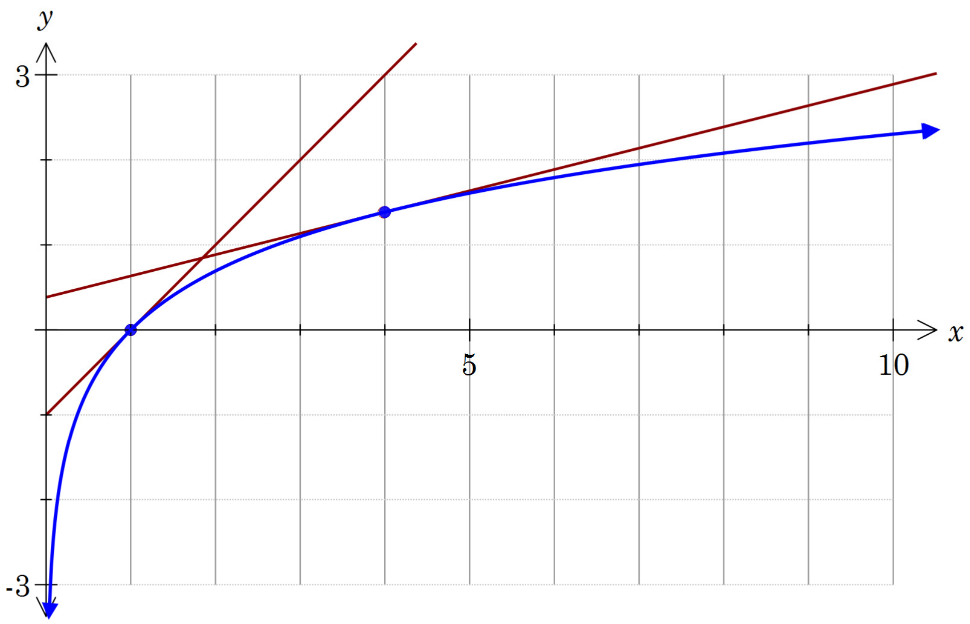
b) $\sim5.0\times10^{-11}$

c) ~0.2



Activity 27 Derivative of $\ln(x)$

1. & 2.



3. At $x = 1$ gradient 1 and at $x = 4$ gradient = $\frac{1}{4}$

4.

x	0.5	1	2	3	4	5
y	–0.693	0	0.693	1.099	1.386	1.609
$\frac{dy}{dx}$	2	1	0.5	0.33	0.25	0.2

5. $\frac{dy}{dx} = \frac{1}{x}$

6. $\frac{dy}{dx} = \frac{1}{x}$

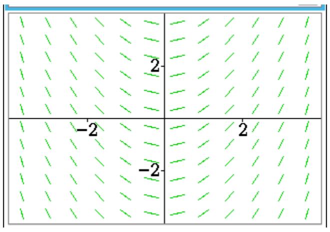
7. Vertical asymptote at $x = 0$

   As $x \to \infty, \frac{dy}{dx} \to 0^{+}$

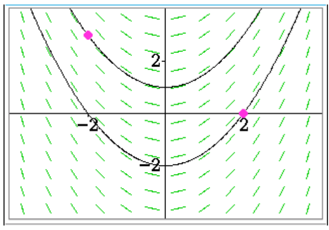
1.

x	-3	-1	0	1	2	3
$\dfrac{dy}{dx}$	-3	-1	0	1	2	3

2. a)

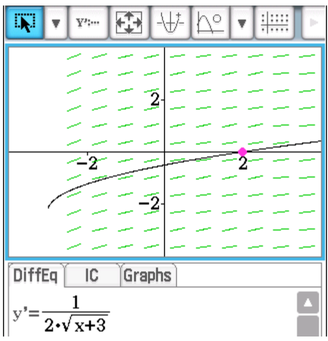


b)

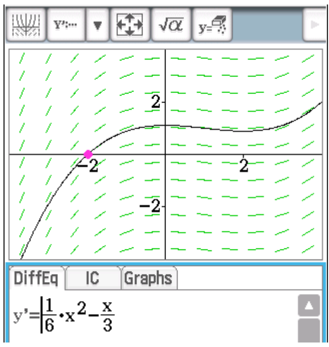


c) $y=\dfrac{1}{2}x^2-2,\quad y=\dfrac{1}{2}x^2+1$

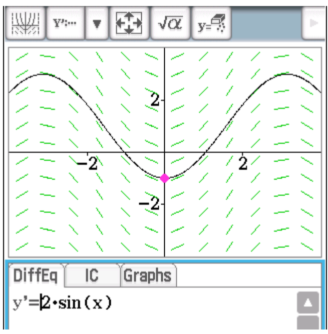
3. a)



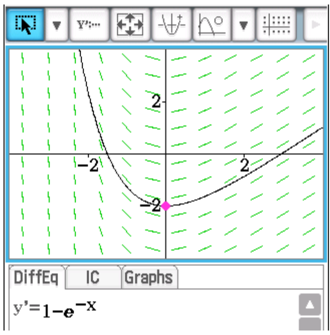
b)



c)



d)



4. a) $y=\sqrt{x+3}-1$

b) $y=\dfrac{1}{2}x^3-\dfrac{2}{3}x^2+1$

c) $y=-2\cos x+1$

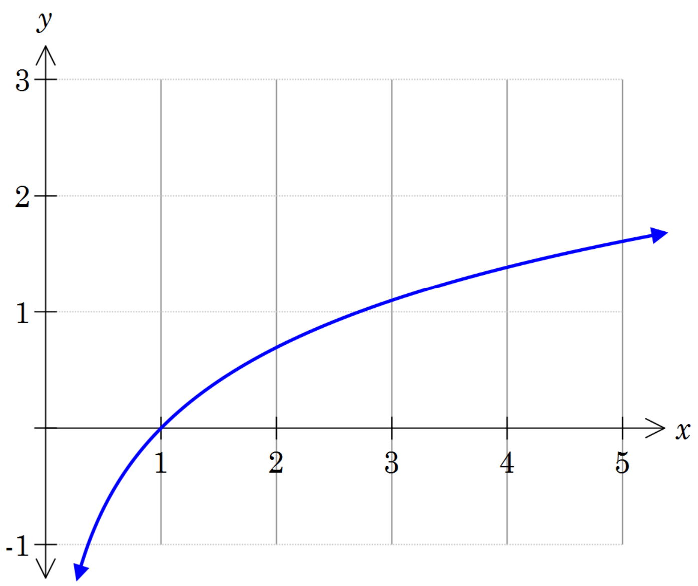
d) $y=x+e^{-x}-3$

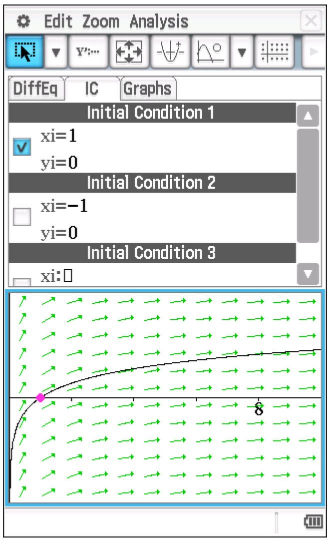
Activity 29 Integral of $1/x$

1.

x	0.5	1	2	3	4	5
$\frac{1}{x}$	2	1	0.5	0.33	0.25	0.2

2. Lines on graph will match ClassPad screenshot.



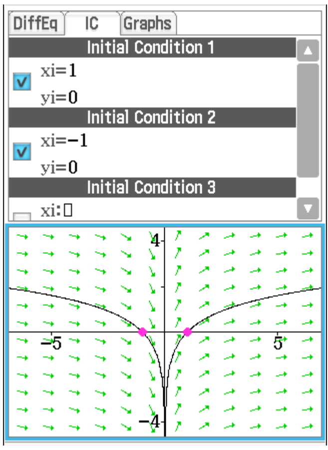


3.

a) Vertical asymptote at $y = 0$, x-intercept at (1, 0)

b) $\ln x$

c) See screen shot



d) $\int\left(\frac{1}{x}\right)dx = \ln(x) + c,\ x > 0$ and $\int\left(\frac{1}{x}\right)dx = \ln(-x) + c,\ x < 0$

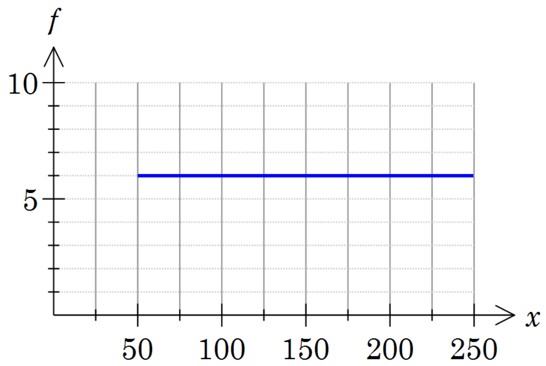
Activity 30 Uniform distribution

1.

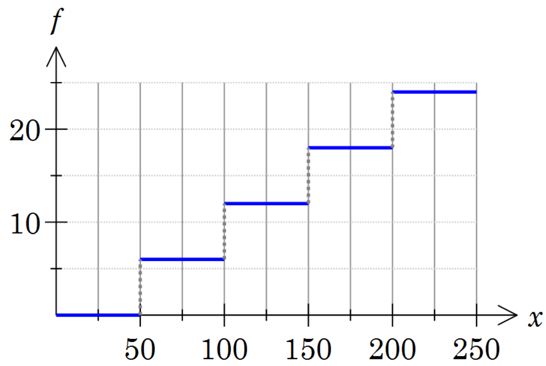
Interval	Cumulative frequency
≤ 49	0
≤ 99	6
≤ 149	12
≤ 199	18
≤ 249	24

2.

Histogram



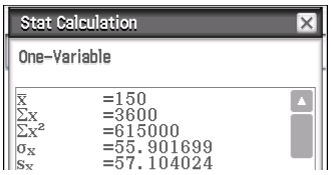
cumulative frequency ogive



3.

a) 150

b) 55.9



4.

a) $\frac{6}{24} = \frac{1}{4}$

b) $\frac{10}{200} = \frac{1}{20}$

The uniform distribution spans 200 m and the interval of interest is 10 m wide.

c) $\frac{1}{200}$

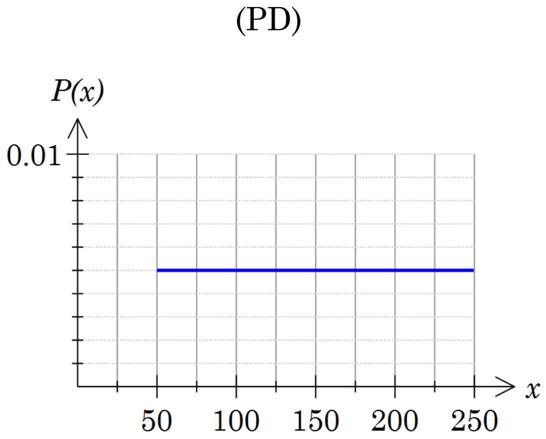
d) $\frac{1}{200}$

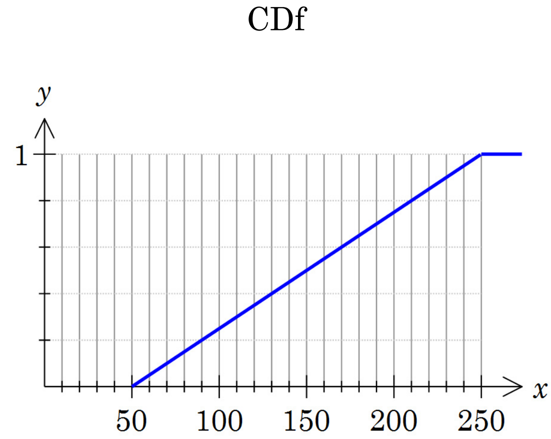
e) $\frac{0.1}{200} = \frac{1}{2000}$

f) 0

g) $\frac{155}{200} = \frac{31}{40}$

5.





a) $P(X = x) = 0.005; 50 \le x \le 250$

b) $C(X < x) = \begin{cases} 0 & x \le 50 \\ 0.005(x-50) & 50 < x \le 250 \\ 1 & x > 250 \end{cases}$

c) $C(X < 205) = 0.775$

The probability of a student selected at random covering less than 205 m is 0.775.

6. Answers will vary.

Activity 31 Calculating with continuous random variables

1.

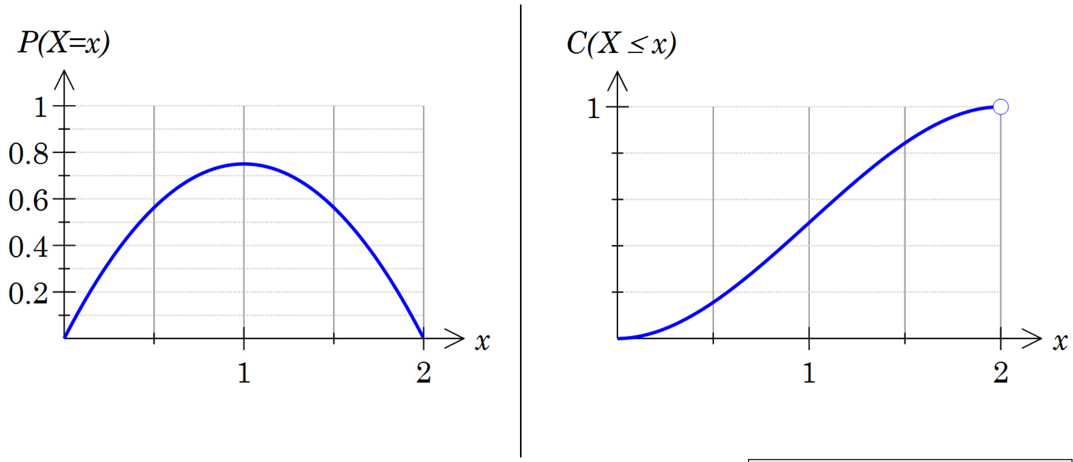
a) $k = 0.75$

b) 1

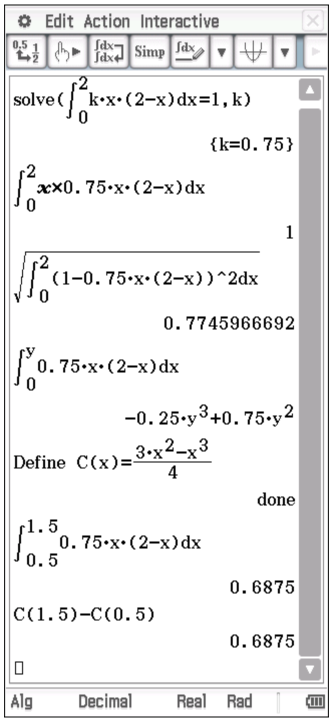
c) 0.447 (3 d.p.)

d) $\dfrac{3x^2 - x^3}{4}$

e)



f) 0.6875



2.

a) 1

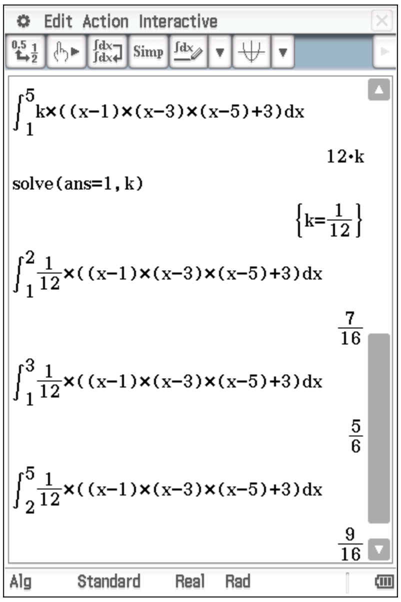
b) $k = \dfrac{1}{12}$

c)

(i) $\dfrac{7}{16}$

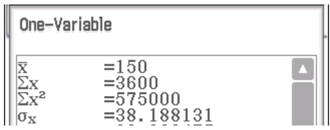
(ii) $\dfrac{5}{6}$

(iii) $\dfrac{9}{16}$



Activity 32 Non uniform continuous random variables

1.

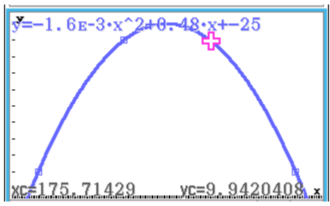


a) $\bar{x} = 150, \quad \sigma = 38.2$

b)

Interval	Cumulative frequency
≤ 99	2
≤ 149	12
≤ 199	22
≤ 249	24

c)



(i) $-0.0016x^2 + 0.48x - 25$

(ii) 67.08, 232.9

(iii) 1216 units²

(iv) $-1.31 \times 10^{-6} x^2 + 0.000395x - 0.0206, \quad 67.08 < x < 232.9$

(v) $-4.39 \times 10^{-7} x^3 + 1.97 \times 10^{-4} x^2 - 0.0206x + 0.623, 67.08 < x < 232.9$

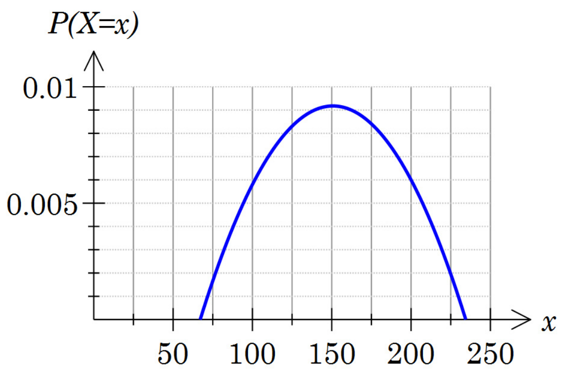
d)

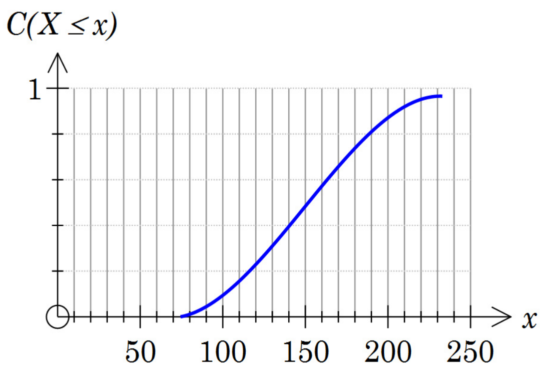
(i) 0.097

(ii) 0.11

2.

a)





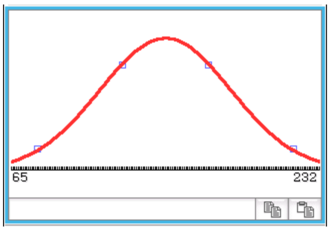
b)

(i) 150 (ii) 1375 (iii) 37.1

3. The domain doesn't extend to the boundaries of the intervals. It matches the mean and standard deviation closely.

4.

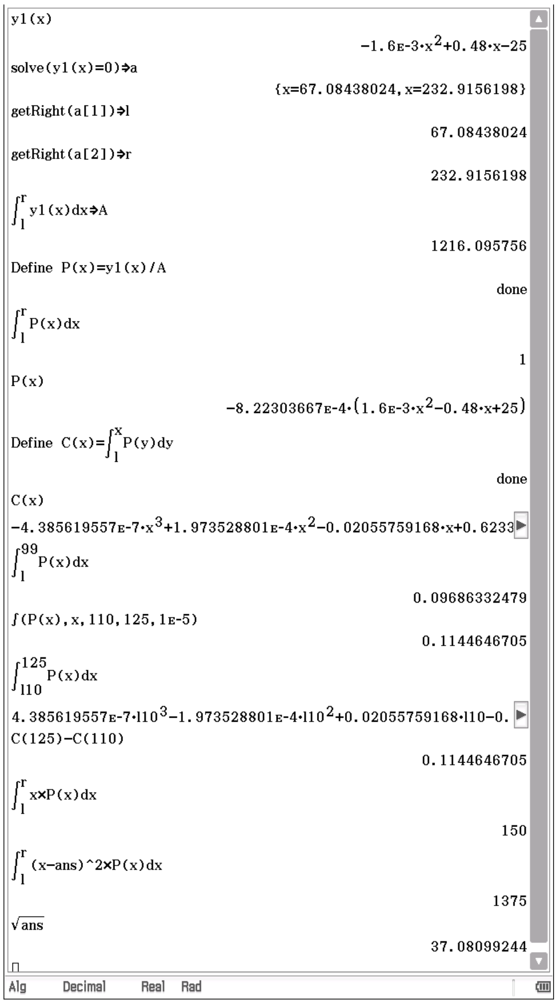
a)



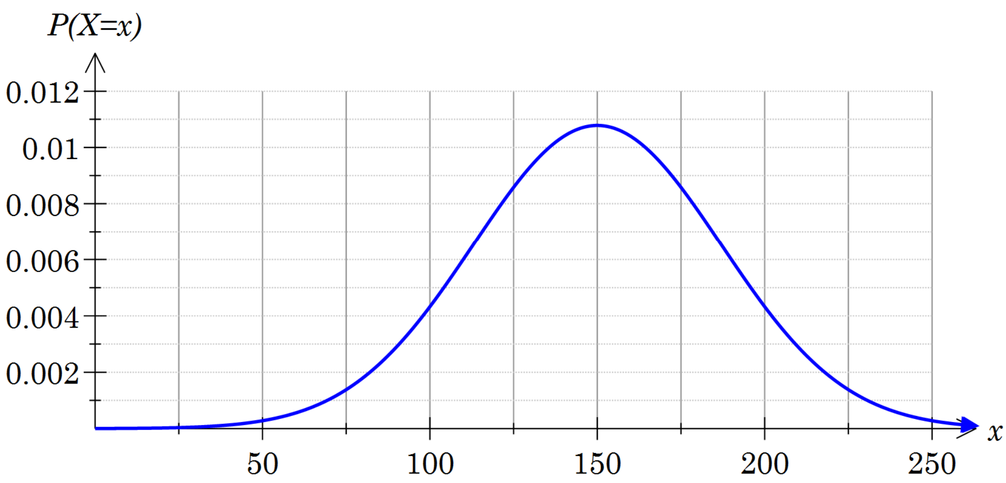
b) It is a bell-shaped curve symmetrical about the mean of 150.

c)

x	$P(X=x)$
75	0.0015
90	0.0030
105	0.0052
120	0.0077
150	0.01
180	0.0077
195	0.0052
210	0.0030



5. a)



a) Values are the same (at least as far as accuracy of estimation from Trace allows).

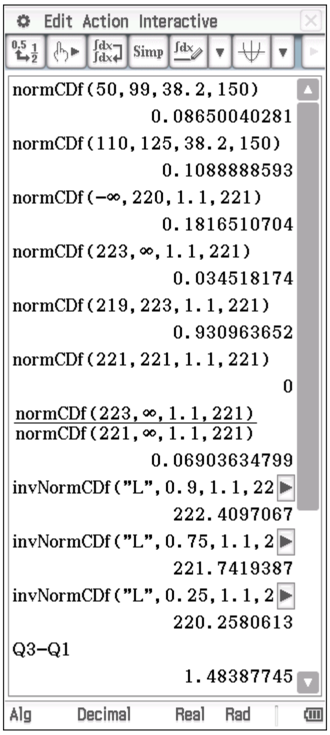
b) The same

Activity 33 Normal CD

1.

a) 0.0865

b) 0.109



2.

a)

(i) 0.182

(ii) 0.0345

(iii) 0.931

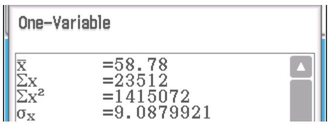
(iv) 0

(v) $\dfrac{0.0345}{0.5} = 0.069$

b) 222.4 mm

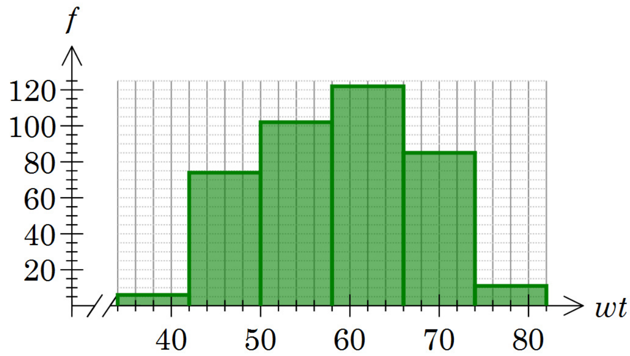
c) 1.48 mm

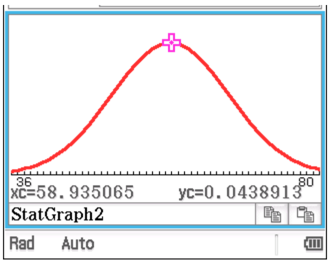
3.



a) $\bar{x} = 58.8 \quad \sigma = 9.09$

b) The normal distribution is an appropriate model because the distribution is approximately bell-shaped.





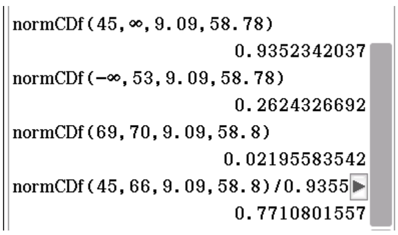
c)

(i) 0.936

(ii) 0.262

(iii) 0.022

(iv) 0.77

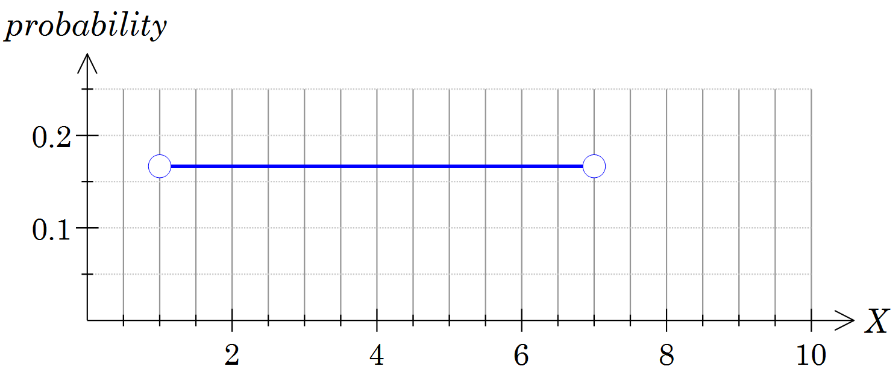


Activity 34 Continuous distributions

1. Examples will vary

Statement	Examples	Description of command
rand	Answers will vary	Generates a 10 digit decimal between 0 and 1.
rand(1,6)	"	Generates an integer between 1 and 6, inclusive.
1+6×rand()	"	Generates a decimal between 1 and 7.
int(1+6×rand())	"	Generates an integer between 1 and 6, inclusive.
int(7 – 6×rand())	"	Generates an integer between 1 and 6, inclusive.

2. a)

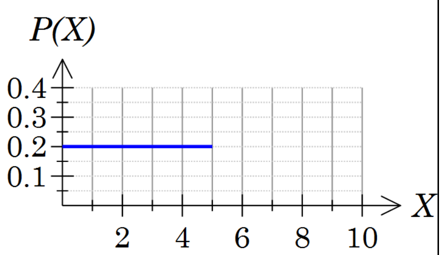


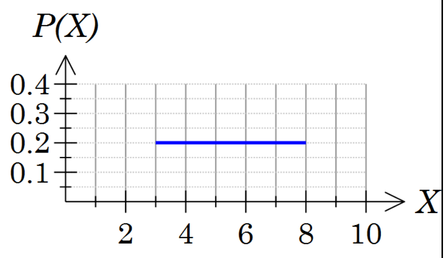
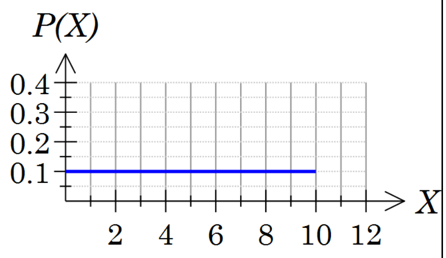
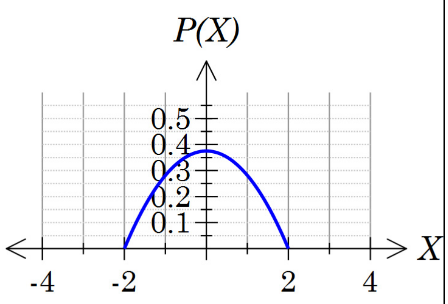
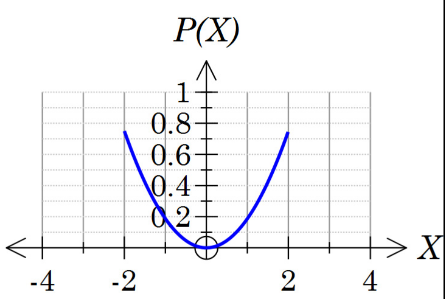
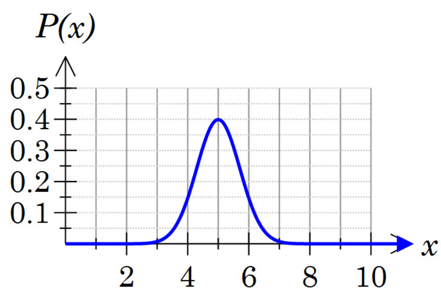
b) 0.25

c) 0.25

d) $\bar{x} = 4$

3.

PDf	Graph	Mean	S.D.
$P(X) = \begin{cases} 0.2 & 0 \le X \le 5 \\ 0 & \text{elsewhere} \end{cases}$		2.5	1.44

$P(X)=\begin{cases}0.2 & 3\le X\le 8\\ 0 & \text{elsewhere}\end{cases}$		5.5	1.44
$P(X)=\begin{cases}0.1 & 0\le X\le 10\\ 0 & \text{elsewhere}\end{cases}$		5	2.89
$P(X)=\begin{cases}\dfrac{12-3X^2}{32} & -2\le X\le 2\\ 0 & \text{elsewhere}\end{cases}$		0	0.89
$P(X)=\begin{cases}\dfrac{3X^2}{16} & -2\le X\le 2\\ 0 & \text{elsewhere}\end{cases}$		0	1.55
$P(X)=\dfrac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(X-5)^2}$		5	1

4.

a) The distribution is shifted 3 units so the mean is increased by 3 and there is no change to the spread.

b) A uniform distribution with twice the range, i.e. twice the spread and standard deviation.

c) More data is close to the middle in part d) hence a smaller spread.

5.

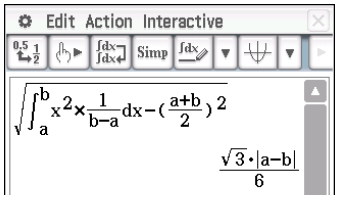
$$\sigma = \sqrt{\int_a^b x^2\left(\frac{1}{b-a}\right)dx - \mu^2}$$

$$= \sqrt{\left[\frac{1}{3}\frac{x^3}{b-a}\right]_{x=a}^{x=b} - \left(\frac{a+b}{2}\right)^2}$$

$$= \sqrt{\left(\frac{1}{3}\frac{b^3}{b-a}\right) - \left(\frac{1}{3}\frac{a^3}{b-a}\right) - \left(\frac{a+b}{2}\right)^2}$$

$$= \frac{\sqrt{3}(b-a)}{6}$$

$$= \frac{b-a}{\sqrt{12}}$$

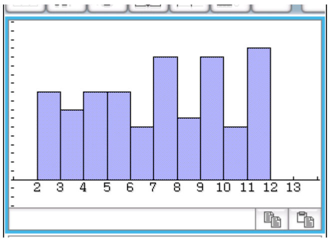


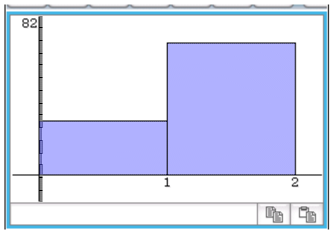
Activity 35 Simulating random samples

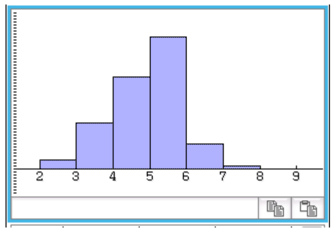
1.

ClassPad expression	Distribution	Range (may be approximate)
2+10×rand()	Uniform	2 – 12
intg(rand()+0.6)	Bernoulli	0 , 1
invNormCDf(rand(),1,5)	Normal	Approx 95% between 3 and 7

2. a) Answers will vary







b) There is significant variation between samples

Approximately a uniform distribution Mean ≈ 7 σ ≈ 2.9	In the long run 1 occurs 60% of the time Mean ≈ 0.6 σ ≈ 0.5	A normal distribution, bell shaped curve Mean ≈ 5 σ ≈ 1

c) Here is an example showing the sort of variation and consistency you might expect in your own samples.

	List 1		List 2		List 3	
Trial	Mean	S.D	Mean	S.D	Mean	S.D
1	7.23	2.97	0.59	0.49	5.03	0.96
2	6.92	2.72	0.62	0.487	4.98	0.924
3	6.99	2.93	0.5	0.50	5.05	1.01
4	7.19	3.05	0.55	0.5	4.99	1.10
5	7.11	2.89	0.55	0.5	4.98	1.06

d) The variation in the means is quite small compared to the variation in the shape of the distributions.

Activity 36 Sample proportions

1. Answers will vary slightly.
2. Tables will vary.
   The sample proportion is close to the population proportion. The standard deviation of the sample proportions is about half the standard deviation of the population proportion. As the number of samples increases, variation of the sample proportions deviate less from the population proportion.
3. Tables will vary.
   The sample proportion is close to the population proportion. The standard deviation decreases as the sample size increases.
4. Tables will vary.
   The sample proportion is close to the population proportion. The standard deviation varies. It is however reasonably consistent with the formula.
   As the number of samples increases, variation of the sample proportion deviates less from the population proportion.
5. The results are reasonably consistent with the assertion that the distribution of $\hat{p}$ is approximately normal with mean p and standard deviation $\sqrt{\frac{p(1-p)}{n}}$ irrespective of the distribution; n is the sample size.

Activity 37 Confidence intervals for proportions

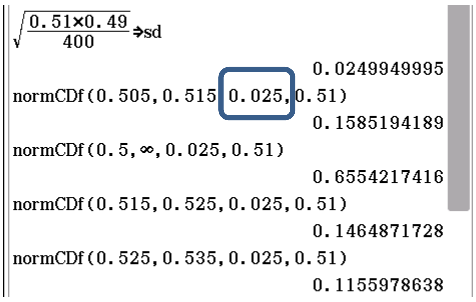
1. $p = 0.51\ ,\ \sigma = \sqrt{\dfrac{0.51 \times 0.49}{400}} = 0.025$ (3 d.p.)

   a) 15.9% of the time

   b) 65.5%

   c) 14.6% (or 14.7% if exact value used for standard deviation. 0.025 is a rounded value)

   d) 11.6%

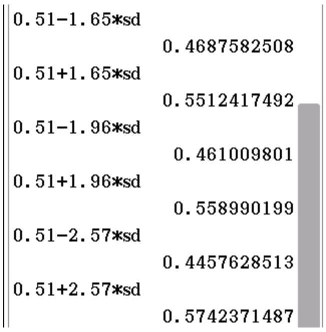


2.

   a) $0.469 \le p \le 0.551$

   b) $0.461 \le p \le 0.559$

   c) $0.446 \le p \le 0.574$

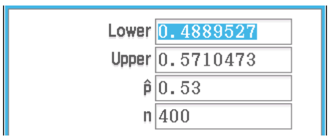


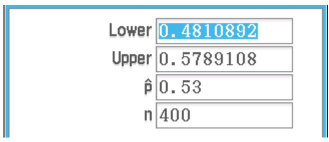
3.

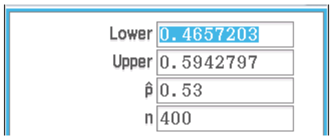
   a) $0.489 \le p \le 0.571$

   b) $0.481 \le p \le 0.579$

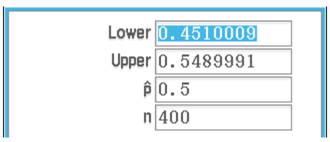
   c) $0.466 \le p \le 0.594$

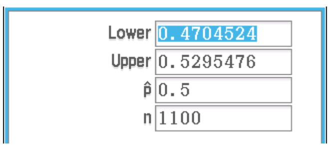


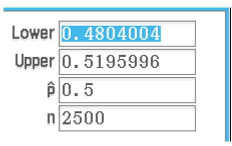


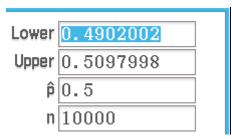


4. Margin of error is based upon a 95% confidence interval, e.g. for a sample of 400 people 95% of the time the population proportion will be within 5% of the sample proportion.









Activity 38 Looking at limits

1. $y\big|_{x=0}$ is not defined. The domain for $y = \dfrac{\sin x}{x}$ excludes 0.

2.

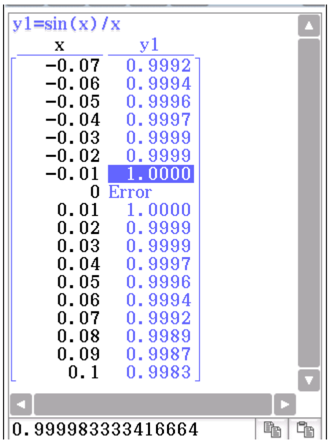
   a) $x \to 0^-, f(x) \to 1$ from below

   b) $x \to 0^+, f(x) \to 1$ from below

   Note: $f(x)$ is an even function, $f(-x) = f(x)$

3.

a)



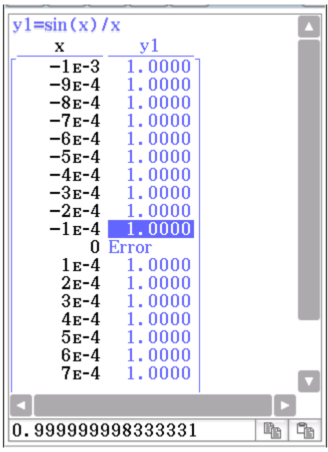
y1=sin(x)/x

x	y1
−0.07	0.9992
−0.06	0.9994
−0.05	0.9996
−0.04	0.9997
−0.03	0.9999
−0.02	0.9999
−0.01	1.0000
0	Error
0.01	1.0000
0.02	0.9999
0.03	0.9999
0.04	0.9997
0.05	0.9996
0.06	0.9994
0.07	0.9992
0.08	0.9989
0.09	0.9987
0.1	0.9983

0.999983333416664

The value is getting closer and closer to 1.

b)



y1=sin(x)/x

x	y1
−1E−3	1.0000
−9E−4	1.0000
−8E−4	1.0000
−7E−4	1.0000
−6E−4	1.0000
−5E−4	1.0000
−4E−4	1.0000
−3E−4	1.0000
−2E−4	1.0000
−1E−4	1.0000
0	Error
1E−4	1.0000
2E−4	1.0000
3E−4	1.0000
4E−4	1.0000
5E−4	1.0000
6E−4	1.0000
7E−4	1.0000

0.999999998333331

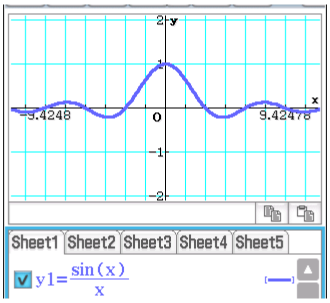
The y-value is getting even closer to 1 but is still less than 1.

4. $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$

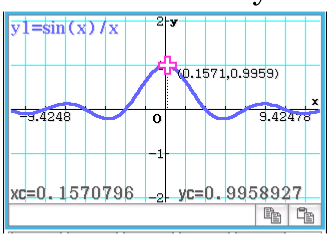
5. The values become closer to each other and the ratio approaches 1.

6. The values become closer to each other and the ratio approaches 1.

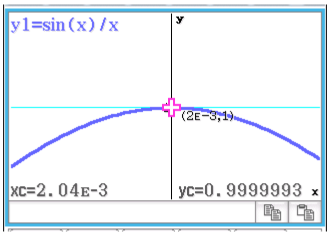
7. The y-value is approaching 1. It looks as if it is 1 but the function is not defined at $x = 0$



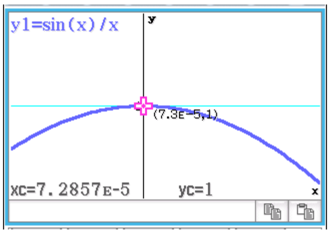
8. Answers will vary 0.9958927 in the screen shot below.



9. Answers will vary, in the screenshot below y = 0.9999993.



10. Answers will vary. It is possible to zoom in sufficiently so that the value displayed is rounded to 1.



By zooming in we can get as close to 1 as we like by making x sufficiently close to 0.

11. 1. It seems we can get as close as we like but never exceed 1.

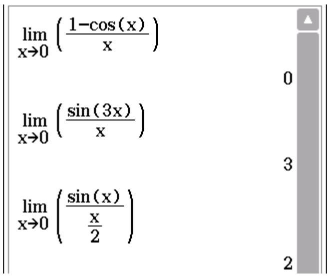
12.

a) 0

b) 3

c) 2

13.



Activity 39 Sine of x on x

1. a) The lines get closer together, and the arc BC becomes more linear and close to vertical. As C gets closer to B, the length of the arc becomes closer to the length of the perpendicular.

   b) Answers will vary depending upon where C is positioned. The table is indicative of what one might expect:

$\overline{CD}$	Arc $\overset{\frown}{BC}$	$\frac{\overline{CD}}{\overset{\frown}{BC}}$
0.98	1.06	0.92
0.5	0.51	0.98
0.225	0.22583	0.99631
0.04250	0.04251	0.99987
0.015	0.015	0.99998

   c) The ratio is increasing and approaching 1.

2. a)

   (i) $CD = \sin\theta$

   (ii) $\overset{\frown}{BC} = \theta$

   (iii) The ratio of CD : BC is $\frac{\overline{CD}}{\overset{\frown}{BC}} = \frac{\sin\theta}{\theta}$

   b) $\frac{\overline{CD}}{\overset{\frown}{BC}} = \frac{\sin\theta}{\theta} < 1$

3. The ratio decreases and is greater than 1. It appears to approach 1.

4. a)

$$\tan\theta = \frac{\overline{CE}}{AC}$$
$$CE = AC\tan\theta$$
$$= \tan\theta$$

   b) The arc is shorter than the tangent.

   c) $\frac{\overline{CE}}{\overset{\frown}{BC}} = \frac{\tan\theta}{1\times\theta}$ (arclength $= r\theta$)

   d)

$$\frac{\overline{CE}}{\overset{\frown}{BC}} = \frac{\tan\theta}{\theta} > 1$$
$$\frac{\sin\theta}{\theta\cos\theta} > 1 \qquad \text{as } \tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$\frac{\sin\theta}{\theta} > \cos\theta$$

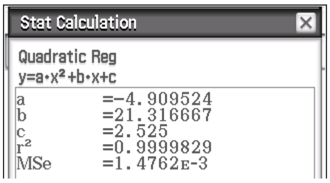
   e) When $\theta = 0, \cos\theta = 1$

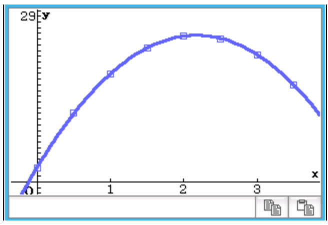
   f) $\therefore \lim_{\theta\to 0}\frac{\sin\theta}{\theta}$ cannot be less than 1.

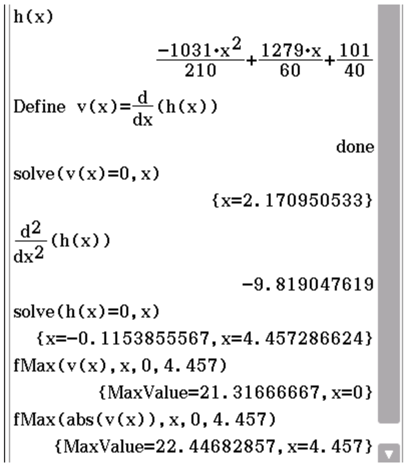
5. $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$ cannot be greater than 1 from Q2 b) and can not be less than 1 from Q4 f). So the limit must be 1.

Activity 40 Modelling motion

1\. a) $h(t) = -4.9t^2 + 21.3t + 2.53$







b) $v(t) = \dfrac{dh}{dt}$

$= -9.8t + 21.3$

c) 2.17 s

d) $a(t) = \dfrac{dv}{dt} = -9.8$

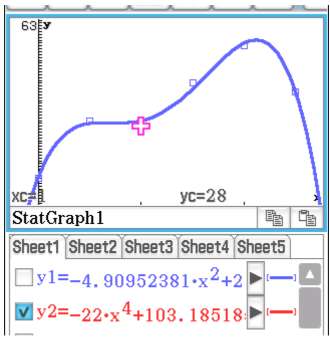
e) 21.3

f) 22.4 m/s after 4.457 s

2\. a) $s(t) = ut + \dfrac{1}{2}at^2$

$v(t) = \dfrac{ds}{dt} = u + at$

b) $a(t) = \dfrac{dv}{dt} = a$



3\. a) $x(t) = -22t^4 + 103t^3 - 152t^2 + 90.5t + 10.2$

b) $t = 0.63,\ 0.78,\ 2.11\ s$

c) 2.74 s

d) $0.63 \le t \le 0.78,\ 2.11 < t < 2.74$

e)

(i) 2.74 s

(ii) 0.70, 1.65 s

(iii) 1.17 s